Machines and Superstars: Technological Change and Top Labor Incomes*

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Abstract

This paper develops a model of hierarchical production organizations to study the effects of technological change on income distribution, with a focus on top labor incomes. The model features workers with different skill levels who interact with machines. The complexity of automated tasks determines whether machines augment or substitute for workers. Two main findings emerge: First, if machines only perform sufficiently simple tasks, they augment low-skilled workers. Consequently, technological advances decrease income concentration at the top by raising low-skill wages more than high-skill wages. Second, if the task complexity of machines surpasses a certain threshold, then machines substitute for low-skilled workers but augment high-skilled workers. As a result, income concentration rises as gains are greatest for the most skilled workers, amplifying the "superstar effect." Lastly, I examine the implications of future AI systems automating managerial functions performed by high-skilled workers. I find that AI managers can reduce income inequality by augmenting low-skilled workers and substituting for high-skilled workers, with the largest gains for the least skilled workers. Overall, the model shows how the complexity of automated tasks determines the effects of technology on income distribution. The results provide insights into diverging trends in top income shares before and after the 1980s, as well as implications of AI for future income inequality.

Keywords: technological change, top labor incomes, production hierarchies, artificial intelligence

JEL Codes: D20, D33, O33

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1 Introduction

A notable change in the U.S. labor income distribution over the past four decades is that the top earners have experienced a faster income growth than the rest of the economy (Atkinson, Piketty, and Saez, 2011), while those in the lower parts of the distribution have under-performed (Acemoglu and Autor, 2011). As a result, the U.S. economy witnessed a divergence between top earners and the rest, indicating that the gains from economic growth have been concentrated. Evidence suggests that technology is a major driving force behind the trend.[1]

In this paper, I build a model of hierarchical production organizations to examine the effects of technological change on income distribution with a focus on top incomes. The model builds on Garicano and Rossi-Hansberg (2006) and Antràs, Garicano, and Rossi-Hansberg (2006), featuring workers and machines. Workers differ in skill levels. Production requires solving problems that differ in complexity, where problems can be interpreted as work tasks. Workers spend time on production and their skill levels determine the maximum complexity of problems they can solve. Similarly, machines also differ in the maximum complexity of problems they can solve, which I refer to as the complexity of machines.

Workers and machines form hierarchical organizations with multiple layers to efficiently use skills. Workers solve relatively simple problems and pass complex problems to their managers. Those with relatively high skill levels become managers who supervise workers. Thus, organizations are hierarchical in the sense that they consist of layers and higher layers solve more complex problems. Machines also solve relatively simple problems and pass the unsolved problems to the upper layer. However, machines cannot supervise workers or other machines.

To examine the effects of technological change on income distribution, I consider an increase in the maximum complexity of machines. Specifically, I model technological change as the introduction of new machines that can solve more complex problems than the existing machines. First, I show that if machines only solve sufficiently simple problems, then organizations have three layers with machines in the bottom and workers in the middle. In this case, workers delegate

[1] Acemoglu and Restrepo (2022) is a recent work on the stagnation of middle and low incomes, and rising wage premium for skilled workers. Acemoglu (2002) and Aghion and Howitt (2008) provide a review of the literature on rising wage inequality. For evidence on top earners' increased income, see Piketty, Saez, and Zucman (2018). Kaplan and Rauh (2010) examine specific high-income occupations including top executives, investors, and lawyers. They conclude that evidence supports theories of skill-biased technological change and superstars.

simple problems that machines can solve and focus on more complex problems. Also, I find that technological change raises workers' wages more than managers' wages. The most skilled managers can even experience a decline in their wages. Intuitively, technological change allows workers to employ more advanced machines and solve more problems, which tends to raise workers' wages. On the other hand, managers gain less because they do not supervise machines directly and the skill levels of their subordinates do not increase.

On the contrary, if the maximum complexity of machines passes a certain threshold, then organizations have two layers with machines and workers in the bottom layer supervised by managers. In this case, machines compete with workers for managers. Technological change increases managers' wages and reduces workers' wages. Moreover, more skilled managers benefit more than less skilled managers. As a result, income distribution becomes more concentrated at the top. The mechanism behind this "cascading effect" of technology is that the most skilled managers employ the most complex machines. Therefore, these managers benefit directly from technological advances and experience the largest wage increases. Additionally, new machines exert downward pressure on the wages of workers and existing machines. The analysis highlights how the complexity of tasks automated by machines matters for the complementarity between machines and skills. The results also demonstrate how gains from technological change could be concentrated among the top earners.

These results offer insights into the decline of top income shares during the mid-20th century and their subsequent increase since the 1980s. One interpretation is that machines during the former period, exemplified by conveyor belts and type writers, were only capable of very simple tasks. Through the lens of the model, low-skilled workers could directly use these machines and were augmented, but high-skilled workers (i.e. managers) who supervise the low-skilled workers, did not benefit as much. With advances in technology in the following years, machines became capable of many tasks done by low-skilled workers. Examples of such machines are robot arms and personal computers. The introduction of more advanced machines increased the supply of skills that augment high-skilled workers, while substituting for low-skilled workers. My model generates patterns that are qualitatively consistent with the changes in top income shares.

After establishing the main results, I explore the implications of advanced AI systems for managerial functions. Due to the latest progress, AI systems have gained a wide range of new capabilities that allow the automation of cognitive tasks. While some express concerns about higher inequality due to automation (Agrawal, Gans, and Goldfarb (2019), Mindell, Reynolds et al. (2022)), others emphasize the potential equalizing effects of AI (Brynjolfsson, Li, and Raymond, 2023) as AI distributes the knowledge possessed by high-skilled workers. Motivated by these developments, I analyze a setup where machines can supervise workers.[2] To do so, I modify the model so that machines (or algorithms in this context) can supervise workers and substitute for managers.

Another modification of the model is that machines are more efficient in supervision than managers. Specifically, I assume that supervision incurs costs in terms of time for both managers and machines, but the cost is lower for machines. Lower supervision costs allow machines to supervise a larger number of workers. In an extreme case, the supervision cost of machines can be negligible, which captures the low inference costs, or fast computation, of modern AI systems.

In the modified setup, I show that if machines have sufficiently small supervision costs, then the number of workers supervised by a machine is no longer limited by the time constraint, and thus time is no longer scarce for machines. As a result, machines supervise all workers below a certain threshold skill level. Moreover, the model yields three results on the effects of algorithmic management on income distribution. First, inequality between workers and managers decreases. Second, inequality among workers declines as the least skilled workers benefit the most from algorithmic management. Third, inequality among the remaining managers rises because the least skilled managers face direct competition with machines.

The main results shed light on the role of technological change in shaping the dynamics of top income shares. In particular, previous research takes two approaches. First, at least since Tinbergen (1956), economists have recognized that technological change may favor certain groups of workers over others. Specifically, technological change is skill-biased if it raises skilled wages, such as those for college graduates, more than unskilled wages.[3] Moreover, recent work shows that advances in automation technologies account for the decline in the wages of a wide range of workers in the US (Acemoglu and Restrepo, 2022).[4] The approach taken in this line of research

[2]Here, I am interested in the effects of wide-spread algorithmic management on income distribution. Note that Lee et al. (2015) define algorithmic management as "software algorithms that assume managerial functions and surrounding institutional devices that support algorithms in practice" and examine ride sharing services such as Uber and Lyft.

[3]See Katz and Murphy (1992) and Acemoglu (1998) for early contributions on skill-biased technological changes.

[4]Acemoglu and Restrepo (2020) estimates the negative effects of robots on employment and wages. There is also literature on routine-biased technological change that focuses on the polarization of the labor markets. See, for

divides workers into fixed groups, for example by education, and mainly aims to understand the changes in relative wages between these groups. While the approach provides insights on rising skill premium and stagnant wages of automated occupations, it does not speak to why gains from economic growth are increasing more for those at higher income levels. Put differently, how are the superstars growing even more successful?

That observation led to the second approach in the literature, which focuses on the superstar effect in the labor market (Rosen (1981), Garicano (2000), Garicano and Rossi-Hansberg (2006)). A main theme is that recent technology disproportionately complements the highest skills, giving rise to "superstars" who are significantly more successful than the rest. However, this approach does not speak to the divergence, since the 1980s, between top incomes and lower incomes, mentioned earlier in the introduction. Moreover, the superstars literature does not account for the potential role of technology during the mid-20th century when the growth of top income was outpaced by that for lower incomes. In this paper, I merge these two approaches and theoretically examine the relationship between technology and top incomes.

The analysis of the implications of algorithmic management highlights a channel through which future AI systems could reduce income inequality by replacing high-skilled workers. The results in this paper speak to the recent evidence that AI, or large language models specifically, benefits low-skilled workers more than high-skilled workers (Noy and Zhang (2023), Brynjolfsson, Li, and Raymond (2023), Peng et al. (2023)). Relatedly, there have been discussions on the possibility of high-skilled workers being replaced by AI (Webb (2020), Hui, Reshef, and Zhou (2023), Agrawal, Gans, and Goldfarb (2023), Felten, Raj, and Seamans (2023)). This paper provides a framework to examine the mechanism that gives rise to an equalizing effect of AI. Moreover, the model shows that advanced AI can lead to much lower labor market inequality by reducing worker differences due to managerial quality or task-specific knowledge.

Additionally, low supervision costs represent a new type of costs that fall significantly in the digital economy (Goldfarb and Tucker, 2019). Previous work has also examined the effects of falling cost of supervision as a result of advances in information and communication technology (ICT) (Garicano and Rossi-Hansberg, 2006). Nonetheless, this paper extends the analysis by considering different supervision costs between managers and machines to study the extreme case

example, Goos, Manning, and Salomons (2014) and related work.

where the supervision cost of machines is arbitrarily small. More specifically, even if the mass of machines is close to zero, so that there is a "single" machine, a significant fraction of workers may be supervised by machines if the supervision cost of machines is also close to zero.[5]

The paper proceeds as follows: Section 2 develops the model. Section 3 contrasts two distinct outcomes on the effects of technology and discusses how the insights apply to the trends in top income shares before and after the 1980s. Section 4 is more focused towards the implications of future AI systems for managerial functions and how top inequality could decrease with the introduction of management by machines. Section 5 concludes by outlining the directions for future work.

2 Model

2.1 Environment and Strategies

Agents and Endowment The economy is populated by a unit mass of agents and lasts one period. There is one good in the economy. Agents differ in their skill levels that are exogenously given. Agents are uniformly distributed on an interval $[1-\Delta, 1]$. Here, $1-\Delta$ is the lowest skill level of agents. Agents are endowed with one unit of time. Agents have a linear utility function over the consumption of the good.

Machines There are machine owners who are endowed with one machine each. Machine owners differ in the complexity of machines, which parallels the skill level of agents. Machine owners are also uniformly distributed on an interval $[\theta, \theta+\phi] \subset [0,1]$ with a distribution function G.

The mass of machine owners is *not* normalized and the density function of the distribution for machines is

$$g(x) = \begin{cases} \mu, & \text{if } x \in [\theta, \theta+\phi] \\ 0, & \text{otherwise} \end{cases}$$

where $\mu \geq 0$. Thus, the total mass of machines existing in the economy is $\mu\phi$, which is not necessarily equal to one. In other words, $\mu\phi$ can be interpreted as the mass of machines relative

[5] In contrast, high supervision costs of managers limit their scope of supervision.

to the agents. Note that if $\mu = 0$ then the model collapses to the standard setup of hierarchical organizations without machines.

The parameters θ and ϕ are not restricted as long as the interval of machines is a subset of the unit interval. The parameter θ is the skill level of the least skilled machines and $\theta + \phi$ is the skill level of the most skilled machines. In other words, $\theta + \phi$ is the maximum complexity of the problems that machines can solve.[6] Figure 1 illustrates these cases. The first highlighted interval captures "rudimentary" machines that can only solve very simple problems relative to workers. On the other hand, the second interval is "advanced" machines on which machines can solve more complex and a wider range of problems.

Production and Organizations To produce the good, agents must spend time and solve problems. Agents can spend their time either to generate problems or supervise others. It costs one unit of time to generate a unit mass of problems. Problems differ in their levels of complexity and are uniformly distributed from 0 to 1. Agents solve problems with complexity levels lower than their skill levels. The amount of output produced by an agent is determined by the mass of problems solved. Thus, in autarky, an agent with skill x solves problems from 0 to x, and the amount of output is x, which is the mass of problems solved. In short, the skill level of an agent equals the amount of output produced by the agent. I assume machine owners cannot produce in autarky.

Alternatively, agents can form organizations, which can also include machines. Organizations are hierarchical and consist of one layer of managers and lower layers of workers and machines. I assume that an organization can have three layers at most. Also, the mass of managers in each organization is normalized to be one so that there is a "single" manager at the top of the hierarchy. Thus, each manager corresponds to an organization consisting of workers and machines in lower layers.

Similarly with workers, a machine with complexity x generates a unit mass of problems and solves problems from 0 to x, producing output of amount x. However, machines cannot supervise workers or other machines. So machines and workers in the bottom layer generate problems and pass the unsolved ones to their managers. Note that if the intervals $[\theta, \theta + \phi]$ and $[1 - \Delta, 1]$ overlap,

[6] For example, early computer vision techniques allowed computers to only categorize pictures of relatively simple objects but recent advances have led to decreasing error rates and applications to more complex problems in real-world situations such as inventory management and self-driving cars.

then workers and machines on the overlapping region solve exactly the same amount of problems.

As an example, suppose an organization has two layers. A worker with skill $x_1 \in [1-\Delta, 1]$ solves problems $[0, x_1]$, of which mass equals x_1. Then the worker passes on the unsolved problems to the manager in the upper layer. The manager, whose skill is $x_2 > x_1$, analyzes the problems by spending time $h < 1$ per unit mass of problems and instantaneously solves those that are less difficult than x_2.[7] Here, h is the supervision cost in terms of time spent per unit mass of problems by a manager. It is time required for managers to analyze the problems received from workers and communicate the results after analyzing them. Since there is no layer above the manager, the measure of problems solved by the worker together with the manager is x_2. Therefore, the final output is x_2.

It is worth noting the differences between generating problems and supervision. Generating problems can be interpreted as the actual production process or activities that require physical involvement. On the other hand, supervision can be considered purely cognitive in the sense that it is only about providing solutions to unsolved problems without engaging in the production activity itself (Garicano, 2000). Thus, managers spend their time by analyzing and communicating the unsolved problems and not by actually solving the problems because they only provide the solutions to the workers.[8]

Relatedly, the restriction on machines can be interpreted as follows. Machines in this economy are robots that are capable of physical activities but require workers to operate them. Thus, higher values of the parameter ϕ can be interpreted as the improvements in industrial machinery during the previous century, which evolved from conveyor belts to robot arms that assemble complex objects.[9]

Strategies At the beginning of the period, agents decide whether to become workers or managers.[10] Those who become workers earn $w_1(x_1)$ and those who become managers hire workers

[7] Like in the existing models of hierarchical organizations, agents specialize either in production (i.e. generating problems) or supervision in equilibrium. Moreover, Garicano (2000) shows that agents in the upper layer has higher skill levels than those in lower layers in equilibrium.

[8] Examples for the role of workers and managers include call center staff members and their superiors, and research assistants and professors. In the latter example, research assistants do the basic work (i.e. generate problems) and ask their professors about difficult issues they cannot resolve on their own.

[9] Of course, the interpretation of machines is not restricted to industrial robots and physical tasks. Other examples include type writers/word processors and grammar checkers/smart chatbots.

[10] Agents can also choose to produce in autarky. Throughout the paper, I restrict attention to the parameter space where all agents are in organizations so I focus on this case.

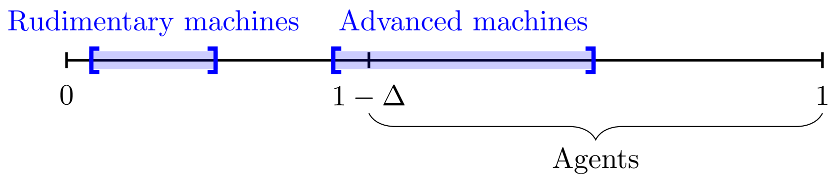


Figure 1: Machines and agents

to supervise and earn $w_2(x_2)$. Agents choose whichever yields higher income. Another way to interpret the environment is that agents trade tasks the market. Workers sort tasks and send the difficult ones to managers. Managers pay $w_1(x_1)$ to workers in return and earn $w_2(x_2)$ themselves by completing the tasks.

In addition to the workers, managers can employ machines as well. A machine owner with machine complexity x receives $w_m(x)$ as a compensation for passing on problems. Here, the subscript m can be either 0 or 1, where 0 indicates the organization has three layers and machines are in the bottom and 1 indicates two layers. In three-layer organizations, all machines are in the bottom layer (layer 0) supervised by workers and are compensated according to a wage function $w_0(\cdot)$.

In two-layer organizations, workers and machines are in the same layer (layer 1) directly supervised by managers. Managers do not distinguish between workers and machines, as long as they solve the same amount of problems (or have the same "skill" levels). Thus, managers form organizations with workers and machines randomly. By the law of large numbers, the fraction of machines within an organization is the same as the economy-wide fraction of machines at the complexity level. Note that if there are two layers then machines and workers solving the same amount of problems face the same wage schedule $w_1(\cdot)$.

The problem of a manager is as follows. If there are two layers in the organization then the manager only chooses the skill level of direct subordinates and solves

$$w_2(x_2) = \max_{x_1, n_1} n_1 x_2 - n_1 w_1(x_1) \tag{1}$$

subject to the time constraints

$$h(1 - x_1) n_1 \leq 1$$

where n_1 is the number of workers hired.[11] Here, managers receive what is left of the total output ($n_1 x_2$) after compensating the workers ($n_1 w_1(x_1)$). Due to the time constraint, agents can spend only up to the time endowment. The amount of time spent can be broken down into three parts. First, a manager spends h to observe each problem. Each worker passes on unsolved problems of mass $1 - x_1$ and there are n_1 workers under the supervision of the manager. Since it is optimal to spend all time endowment, the time constraint holds with equality, which pins down the number of workers

$$n_1 = \frac{1}{h(1 - x_1)}$$

Note that n_1 is increasing in x_1. Intuitively, more skilled workers allow the manager to hire more workers because they require less supervision time from their manager. With the saved time per worker, the manager can supervise a larger group of workers. By substituting the above expression into the objective, the manager's problem can be written as

$$w_2(x_2) = \max_{x_1} \frac{x_2 - w_1(x_1)}{h(1 - x_1)}$$

Panel (a) in Figure 2 illstrates this case where the top circle is the manager supervising workers and machines.

If there are three layers with machines in the bottom layer, then the manager chooses the skill level of workers as well as that of machines.

$$w_2(x_2) = \max_{x_0, x_1, n_0, n_1} n_1 n_0 x_2 - n_1 w_1(x_1) - n_1 n_0 w_0(x_0) \tag{2}$$

subject to the time constraints

$$h(1 - x_1) n_1 n_0 \leq 1$$
$$h(1 - x_0) n_0 \leq 1$$

[11] Here, I assume that managers choose workers of only one skill level. In the following subsection, I show that the assumption is without loss of generality. For a related discussion, see Antràs, Garicano, and Rossi-Hansberg (2006) and their working paper version.

where n_1 is the number of workers as above and n_0 is the number of machines per worker. Thus, the span of control of the manager, or the total mass of problems generated in the organization, is given by $n_1 n_0$. In the three-layer case, the total cost is the sum of compensations paid to workers and machines. As in the two-layer case, the manager's problem can be rewritten by substituting in the time constraints

$$w_2(x_2) = \max_{x_0, x_1, n_0, n_1} \frac{x_2}{h(1-x_1)} - \frac{1-x_0}{1-x_1} w_1(x_1) - \frac{w_0(x_0)}{h(1-x_1)}$$

Panel (b) of Figure 2 depicts this case where the manager directly supervises workers only (middle circles) and workers supervise machines (bottom squares).

It is worth noting that the production function of an organization is supermodular since the total output is given by $x_2/h(1-x_1)$ that take x_1 and x_2 as inputs. Taking the derivatives, it follows that

$$\frac{\partial^2}{\partial x_1 \partial x_2} \left(\frac{x_2}{h(1-x_1)} \right) > 0$$

Intuitively, all managers can produce more if they hire better workers but the increase is larger for better managers than worse managers. Similarly, all workers become more productive if better managers supervise them but the productivity gain is greater for better workers than worse workers.

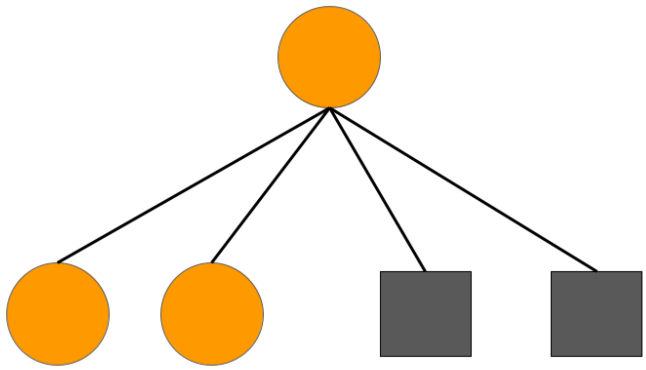
2.2 Equilibrium

2.2.1 Equilibrium Characterization

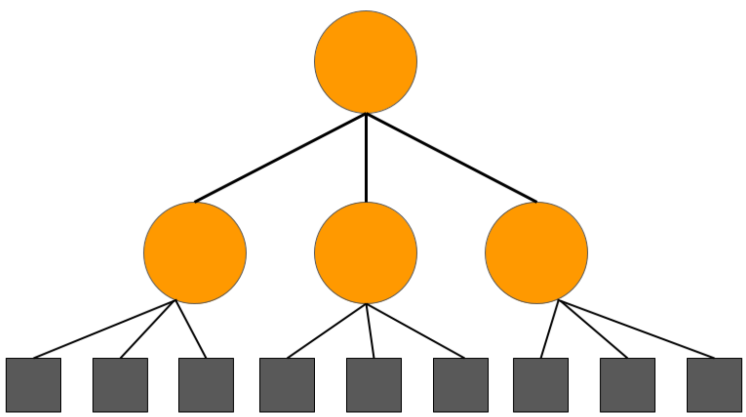
I start with the definition of a competitive equilibrium in this economy.

Definition 1 (Equilibrium)**.** *A competitive equilibrium consists of (i) an allocation of agents between workers and managers, (ii) a set of machines in the market, (iii) the number of layers* $L \in \{2, 3\}$ *in organizations, (iv) wage functions, (v) a mapping from managers to workers such that*

1. *agents maximize their utility given the wages, the assignment function, and threshold skill levels;*
2. *markets clear for all skill levels.*



(a) When machines compete with workers



(b) When machines augment workers

Figure 2: Organizational structure

Note: Upper circles are managers, lower circles are workers, and squares are machines.

Since there are no market imperfections in the economy, a competitive equilibrium is Pareto optimal. Therefore, to obtain the decentralized allocation, it suffices to solve the planner's problem.

The planner matches workers and managers to maximize the total output of the economy. The following lemma shows that each manager hires workers of only one skill level.

Lemma 2. *Each manager supervises workers of only one skill level.*

Proof. See Appendix A.1. □

The intuition behind the proof is that the planner exploits supermodularity to efficiently allocate the time of agents. In a decentralized equilibrium, wages adjust to support the first-best allocation. Thus, I focus on the manager's problems stated in Section 2.1 throughout the paper.

To characterize an equilibrium of this economy, I begin with the labor market clearing conditions. In equilibrium, the labor markets must clear for all skill levels with the supply and demand for workers equalize.

Suppose there are two layers and consider the assignment of managers on an interval $[x_2, x_2 + dx_2]$ to workers on $[x_1, x_1 + dx_1]$. Then the labor market clearing condition requires

$$[f(x_1) + g(x_1)]dx_1 = n_1 f(x_2)dx_2$$

where the left-hand side is the supply of workers as a sum of agents and machines. The right-hand side is the demand for workers, which is the demand from each manager $n_1(x_2)$ multiplied by the number of managers $f(x_2)dx_2$. The demand of manager for workers is $n_1 = 1/h(1 - x_1)$, and thus the condition can be written as

$$f(x_1) + g(x_1) = \frac{1}{h(1 - x_1)} f(x_2) \frac{dx_2}{dx_1}$$

Denote the equilibrium relationship between x_1 and x_2 by the assignment function $x_2 = a(x_1)$. Then the labor market clearing condition, or the assignment equation, is

$$f(x_1) + g(x_1) = \frac{1}{h(1 - x_1)} f(a(x_1))a'(x_1) \quad (3)$$

Equation (3) is a differential equation that pins down the equilibrium assignment function $a(\cdot)$ together with a boundary condition. Notice that the slope of the assignment function is increasing in the supply of workers. Thus, the addition of machines, or a new pool of workers in general, makes the assignment steeper. Intuitively, a steeper assignment function implies that slightly better workers are supervised by much better managers than worse workers. The addition of machines, therefore, increases the difference of workers in their managers' skill levels.

A similar logic applies to the case with three layers. The difference is that the equilibrium assignment function is defined over two separate intervals. Suppose machines on $[x_0, x_0 + dx_0]$ and workers on $[x_1, x_1 + dx_1]$ are matched with managers on $[x_2, x_2 + dx_2]$. Then

$$g(x_0)dx_0 = n_1 n_0 f(x_2)dx_2$$
$$f(x_1)dx_1 = n_1 f(x_2)dx_1$$

where the left-hand side is the supply of machines and workers, and the right-hand side is the demand of managers. Again, substituting the span of control into the equations I have

$$g(x_0) = \frac{1}{h(1-x_1)} f(x_2) \frac{dx_2}{dx_0}$$
$$f(x_1) = \frac{1-x_0}{1-x_1} f(x_2) \frac{dx_2}{dx_1}$$

In this case, the assignment function $a(\cdot)$ maps the set of machines into the set of workers and maps the set of workers into the set of managers. Then the market clearing conditions are

$$g(x_0) = \frac{1}{h(1-a(x_0))} f(a(a(x_0)))a'(a(x_0))a'(x_0)$$
$$f(a(x_0)) = \frac{1-x_0}{1-a(x_0)} f(a(a(x_0)))a'(a(x_0))$$

By dividing the first equation with the second and substituting $x_1 = a(x_0)$, the final expression for the labor market clearing conditions is

$$g(x_0) = \frac{1}{h(1-x_0)} f(a(x_0))a'(x_0)$$
$$f(x_1) = \frac{1-a^{-1}(x_1)}{1-x_1} f(a(x_1))a'(x_1)$$

Note that the above conditions are assignment equations for adjacent layers. The same conditions can be derived with the matching between machines and workers, and workers and managers.

The equilibrium allocation exhibits positive sorting since the slope of the assignment function is always strictly positive. To gain intuition, consider the optimal allocation of this economy. Due to the supermodularity of the production function, the planner wants highly productive managers to be matched with highly productive workers. And in fact, this is also true in the decentralized equilibrium since it coincides with the first-best allocation.[12] Note also that the sorting is strictly positive and so the assignment function is one-to-one as well as onto. That is, it is not optimal for a manager to supervise an interval of workers because it is more efficient to spend all available time only on the most skilled workers.

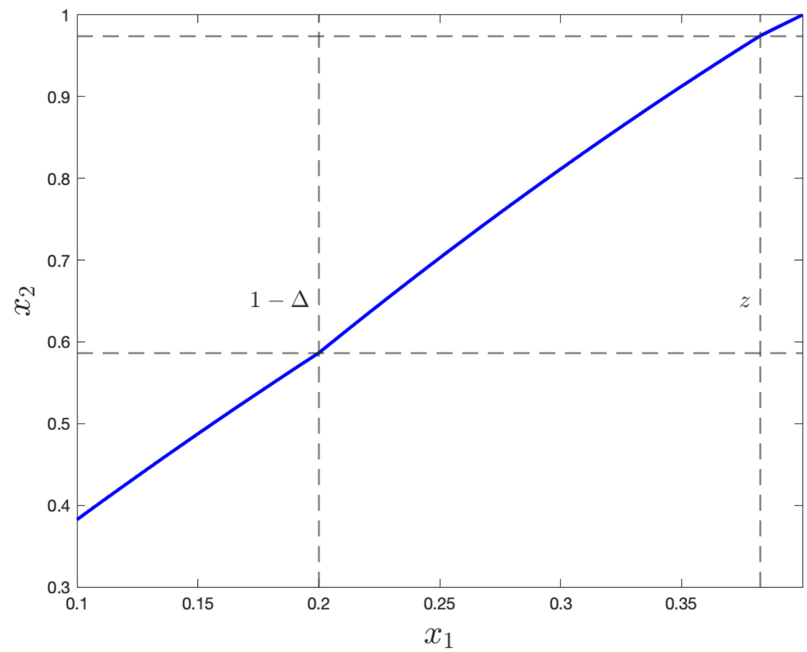
Also, note that the assignment function is concave. The intuition for concavity is that more skilled managers supervise an increasingly larger number of workers and thus more skilled workers are supervised by relatively similar managers. Put differently, the inverse assignment function, which maps managers into workers, is convex. Convexity implies that the difference in worker skill is larger for more skilled managers. The reason is that more skilled managers supervise a large number of workers and slightly less productive managers can only supervise much less productive workers that are left for them.

Wages are such that support the equilibrium assignment of agents. Thus, equilibrium wage functions are determined by the first-order conditions of the manager's problem. In the case with two layers, the first-order condition to (1) is

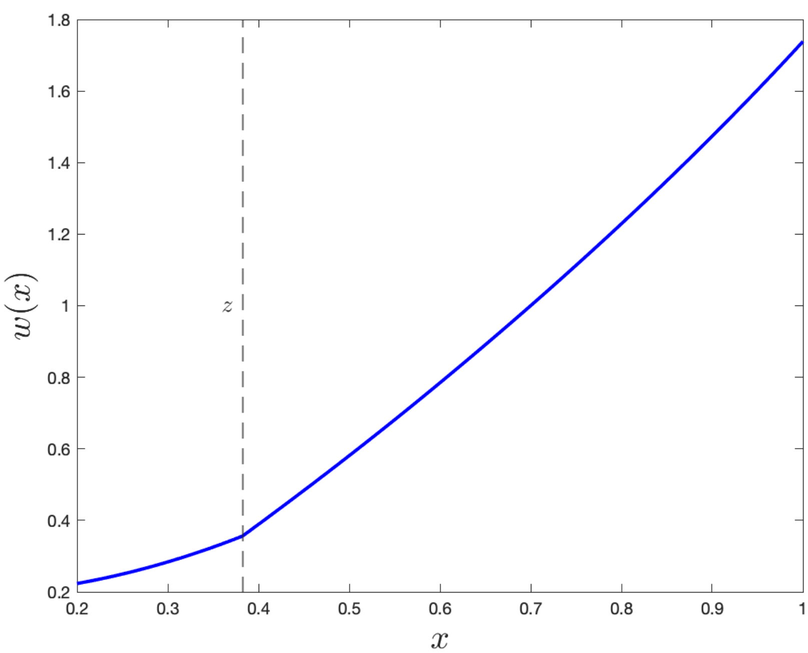
$$w_1'(x_1) = \frac{x_2 - w_1(x_1)}{1-x_1} \tag{4}$$

The left-hand side is the marginal cost of hiring slightly more skilled workers, which is higher wages

[12]As in Garicano and Rossi-Hansberg (2006), the complementarities are reflected in the wages and the markets are complete. Thus the decentralized equilibrium is efficient.



(a) Assignment function



(b) Wage function

Figure 3: Equilibrium assignment and wages

paid to the workers. The right-hand side is the benefit, which is the gains in output as a result of expanding the span of control.

Likewise, the first-order conditions determine the equilibrium wage functions in the three layer case as well. By taking the derivatives with respect to x_0 and x_1, I obtain

$$\begin{aligned}
&[x_1] \quad \frac{x_2}{h(1-x_1)^2} = \frac{1-x_0}{1-x_1} w_1'(x_1) + \frac{1-x_0}{(1-x_1)^2} w_1(x_1) + \frac{w_0(x_0)}{h(1-x_1)^2} \\
&[x_0] \quad \frac{w_1(x_1)}{1-x_1} = \frac{w_0'(x_0)}{h(1-x_1)}
\end{aligned}$$

The left-hand side of the first condition tells that, by hiring slightly more skilled workers, the manager can increase output. As the right-hand side shows, costs increase as well because (i) more skilled workers require higher wages, (ii) the manager can hire more workers, and (iii) more skilled workers supervise more machines.

In the second condition, the benefit of hiring slightly more productive machines is the reduction in the compensation paid to the workers because more productive machines makes each worker more productive and thus the manager may reduce the total number of workers. The cost is again higher wages paid to machines.

Convexity of the wage function reflects the superstar effect in this economy. The income difference between adjacent managers is larger for more skilled managers. Thus, near the upper tail of the income distribution, slightly more skilled managers earn significantly more than less skilled managers. The focus of the comparative static analysis in Section 3 is how this convexity, and thus the superstar effect, depends on technological change captured by an increase in ϕ.

As shown by existing work, such as Garicano and Rossi-Hansberg (2006), the set of agents in each layer is connected in equilibrium. Thus, the allocation of agents is summarized by a threshold skill level z. The following lemma shows that it is indeed the case in the current economy as well.

Lemma 3 (Occupational Choice)**.** *For some z, agents with skill levels higher than z become managers, and those below become workers. Agents exactly at z are indifferent.*

Proof. See Appendix A.2. □

The threshold z is an outcome of occupational choice, which divides workers and managers in equilibrium. With the addition of machines, there is another threshold that characterize the stock of machines available in equilibrium. The following lemma shows that only sufficiently productive machine owners enter the market for problems due to the presence of an entry cost.

Lemma 4 (Entry Threshold of Machines)**.** *Suppose there is a sufficient amount of machines. Then, for some $\theta^* \geq \theta$, only machine owners above θ^* enter the market and those below the threshold do not enter. Machine owners exactly at θ^* are indifferent.*

Proof. See Appendix A.3. □

Lemmas 3 and 4 imply that the sets of workers and machines that participate in the labor market are $[1-\Delta, z]$ and $[\theta^*, \theta+\phi]$. Note that if $\theta^* > \theta$ then the entry condition is binding and thus there are fewer machines in the market than the total amount available.[13]

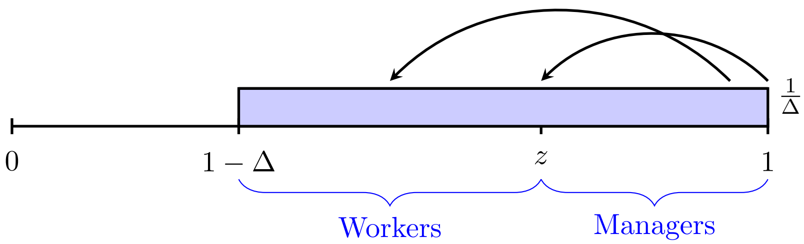
Figure 4 illustrates the assignment of agents and machines given the thresholds. Panel (a) is the assignment of workers and managers, which corresponds to the basic model of production hierarchies developed in, for example, Garicano and Rossi-Hansberg (2006). The arrow from 1 to z is the matching between the most skilled managers and workers. The second arrow is the matching between less skilled managers and workers.

Panel (b) includes machines, which are indicated by the additional rectangles in red. The density of machines is μ and thus the density of all workers and machines on the overlapping region is $1/\Delta + \mu$. Therefore, workers on this region face a greater competition for managers. Note that in the figure, the most skilled managers still supervise the most skilled workers at z.

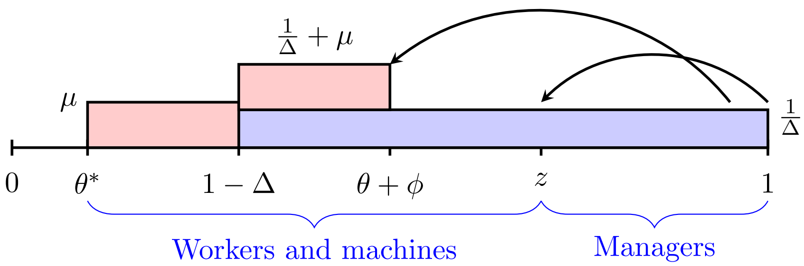
Figure 3 shows an equilibrium assignment and wage functions. Panel (a) is the assignment function for an economy where all organizations have two layers. The figure shows a mapping from the lower layer consisting of workers and machines to the upper layer of managers. Note that the mapping is defined piecewise over the intervals $[\theta, 1-\Delta]$, $[1-\Delta, z]$, and $[z, \theta+\phi]$, on which the function is concave.

Panel (b) of Figure 3 shows the wage function against skill levels. The wage function is continuous at z because of the indifference condition for between workers and managers. Moreover, the wage function is convex for both workers and managers. The reason is the supermodularity of the production function. The marginal product of workers is increasing in skill level, which justifies

[13] The reason for an entry cost is technical. Since machines do not choose occupations, there needs to be a mechanism that ensures that the supply of and demand for machines are met in equilibrium, which is what θ^* does. If θ is sufficiently high then the entry cost becomes small relative to the wages that machines receive and so $\theta^* = \theta$.



(a) Agents only



(b) Agents and machines

Figure 4: Allocation of agents and machines

Note: Agents are blue and machines are red.

the convexity of workers' wages. Also, managers' wages are convex because the output function is convex in the manager's skill level. Intuitively, more skilled managers are able to expand their span of control to a greater extent by hiring more skilled workers and saving their time spent on each worker. Since more skilled managers supervise a larger number of workers, even a small increase in workers' skill turns into a large gain in terms of the span of control. And the equilibrium wage function supports the allocation exhibiting positive sorting.

2.2.2 Existence and Uniqueness

I focus on the cases where all organizations have the same structure, i.e. the same number of layers, in equilibrium. In particular, I restrict my attention to the cases where (i) all organizations have two layers with machines and workers both located in the bottom layer and (ii) all organizations have three layers with only machines in the bottom layer below workers. In each of these cases, machines either complement workers or compete with them for managers as different layers are complementary to each other. Thus, agents who are in separate layers from machines are complemented, linked by the assignment function. On the other hand, if machines are in the same layer as workers, then they raise the supply of workers that managers face and increase the competition for managers.

The following propositions show the existence and uniqueness of an equilibrium that supports two or three layers. Moreover, the results reveals how the supervision cost h and the overall technology level θ determine the equilibrium organizational structure.

Proposition 5 (Equilibrium with Two Layers)**.** *The economy has a unique equilibrium where all orgaziations have two layers with workers and machines in the bottom layer if all machines are sufficiently productive and the supervision cost h falls into some interval $I_h^2 \subset [0, 1]$.*

Proof. See Appendix A.4. □

The proof of Proposition 5 shows that, in order for all organizations to have two layers in equilibrium, the level of technology must be sufficiently high so that managers directly supervise machines. Otherwise, managers may have an incentive to switch to an organization with three layers to delegate the supervision of less skilled workers/machines to more skilled workers.

On the contrary, if machines have sufficiently low skill levels then they are located in the bottom layer below workers, and thus organizations have three layers. The following proposition proves the existence and uniqueness of such an equilibrium.

Proposition 6 (Equilibrium with Three Layers)**.** *The economy has a unique equilibrium where all orgaziations have three layers with only machines in the bottom layer supervised by workers if all machines have sufficiently low skill levels and the supervision cost h falls into some interval $I_h^3 \subset [0, 1]$.*

Proof. See Appendix A.5. □

Unlike in the two-layer case, θ must be sufficiently low for organizations to have three layers. Otherwise, managers may have an incentive to hire machines directly or hire workers only because machines become expensive.

Figure 5 illustrates how changes in parameters θ and h determine whether or not the allocation of interest is an actual equilibrium output. The blue region in the figure is where one of the two allocations described in Section 2.2 is an equilibrium: (i) two layers with machines and workers in the lower layer, and (ii) three layers with machines in the bottom and workers in the

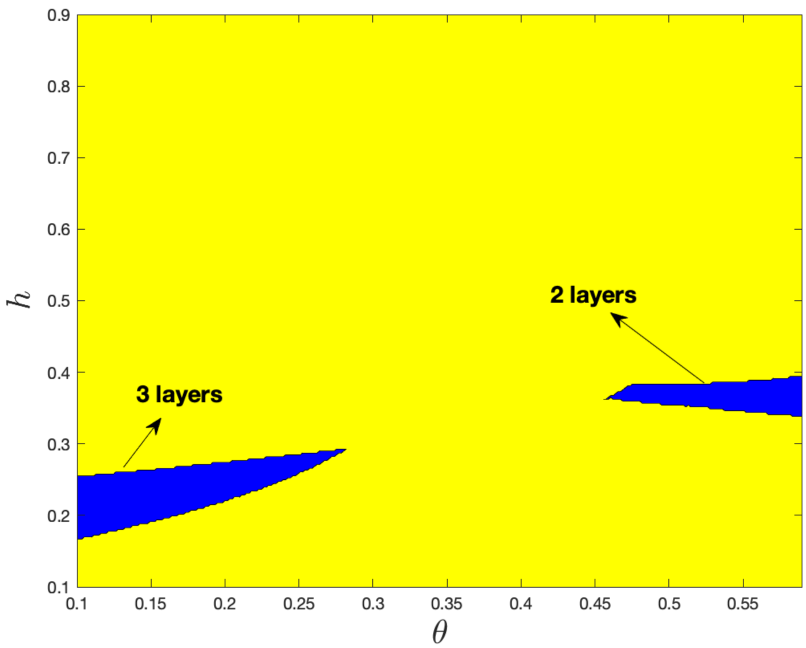


Figure 5: Equilibrium on the (θ, h) space

middle layer. The yellow region is where neither of the allocations is an equilibrium because there are some agents who can profitably deviate from the allocations.[14]

The blue region is separated into two parts. The right part indicates where the allocation with two layers is an equilibrium. To gain intuition, it is worth noting that the right part of the blue region is the intersection of the region where agents do not have the incentive to switch to three layers and the region where there is no self-employment. If the supervision cost h is too low then supervision becomes less costly and thus managers have the incentive to form organizations with three layers. On the other hand, if h is too high then forming organizations becomes too costly and it is more profitable to be self-employed instead of spending time on communication in organizations. Thus, the allocation in Proposition 5 requires h to be in the "goldilocks" zone that rules out both incentives to deviate. Moreover, the equilibrium requires technology level θ to be sufficiently high given the heterogeneity in machines ϕ. Otherwise, the most skilled machines are not matched with the most skilled managers, which is needed for the results in Section 3. Lastly, the right part of the blue region ends at lower values of θ (around 0.45 in the figure). This is because

[14]Also, it is possible that other types of organizations arise in equilibrium. Although it may be interesting to explore various possibilities for the relationship between the parameters and organizations, in this paper I focus on the blue region and the corresponding allocations.

managers can reduce compensation for workers by delegating the easiest problems to machines with low skill levels.

The left part is where three layers are an equilibrium outcome. Again, the equilibrium is in the goldilocks zone where h is neither too high or too low. If h is too high, then it is not efficient to maintain three layers because communication becomes costly. Instead, managers may profitably deviate by switching to two layers. On the other hand, if h is too low, then workers become more efficient in supervising machines and so the demand for machines is too high compared to the existing stock.

3 Technological Change and Top Incomes

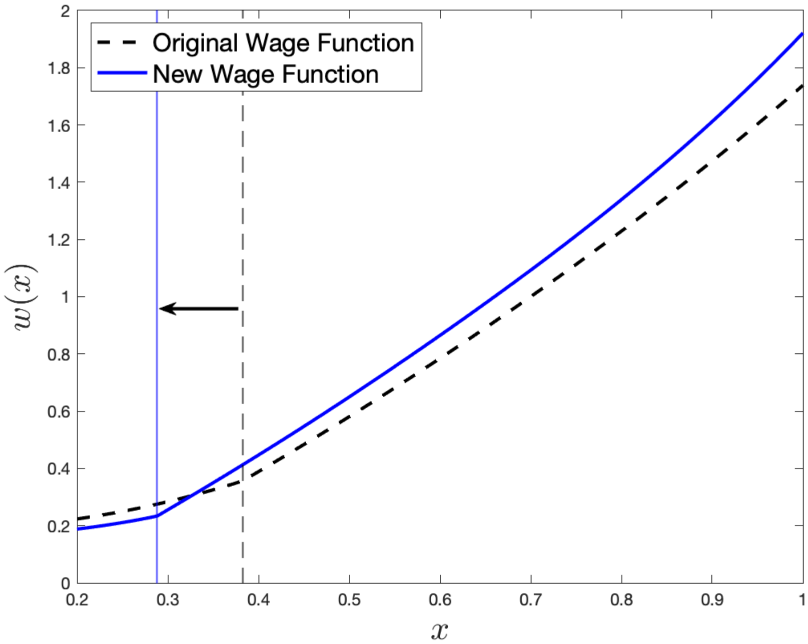
A key takeaway from the previous section is that the complementarity among managers, workers, and machines depends on the organizational structure determined by the level of technology. This subsection examines how technological change may have distinct effects on top incomes depending on whether machines complement workers or managers.

I define technological change as an increase in ϕ, which expands the interval of machines $[\theta, \theta + \phi]$. Technological change introduces new machines that can solve more difficult problems than the existing ones. If workers are in a different layer than machines, workers may benefit from the complementarity in the production technology. On the other hand, if workers are in the same layer as machines, then it may be managers who benefit from technological change while workers experience falling wages due to the increased competition for managers. The next two subsections show that this is indeed the case.

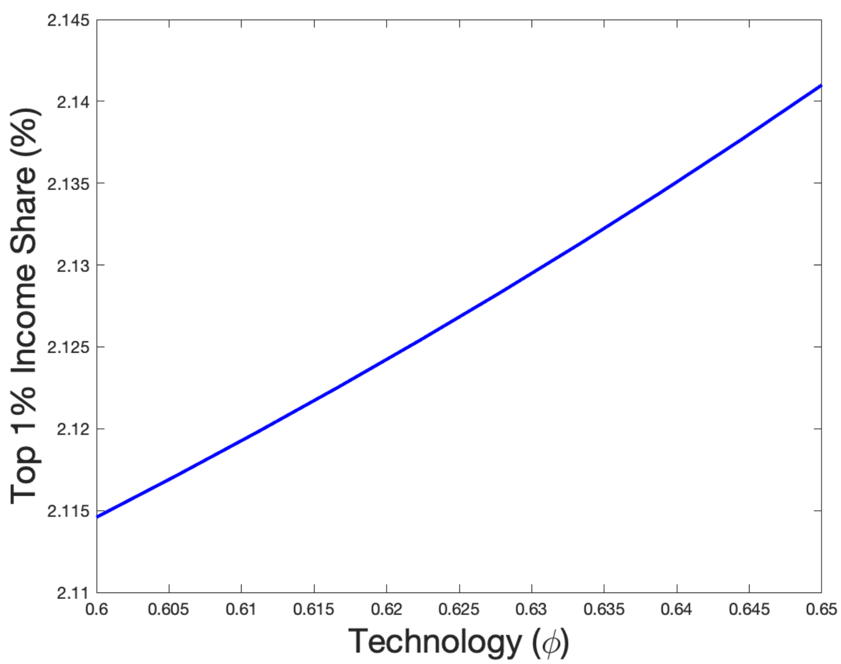
3.1 When Machines Compete with Workers

For the comparative static analysis, I focus on the parameter space to which Proposition 5 applies. Furthermore, I am interested in the allocation where the most advanced machines are matched with the most skilled managers. This is true if $\theta + \phi > z$. That is, if the most advanced machines can solve more difficult problems than any worker, then the most skilled managers employ the most advanced machines due to positive sorting.

In the allocation with two layers, z admits a closed-form expression. By observing the



(a) Changes in wages



(b) Top 1% income share and advances in machines

Figure 6: When machines compete with workers

equilibrium threshold z and the inequality, it follows that $\theta + \phi$ must be sufficiently high given the other parameters. Thus, I impose the following assumption on the parameters θ and ϕ.

Assumption 7. *The maximum complexity of machines is sufficiently high.*

Since machines cannot supervise others, all machines including those on $[z, \theta + \phi]$ are supervised by managers. In particular, the most advanced machines are supervised by the most skilled managers as a result of positive sorting.

Now suppose technological change leads to a rise in ϕ by $d\phi$. A rise in ϕ introduces new machines that have higher skill levels than the existing ones by extension of the interval $[\theta + \phi, \theta + \phi + d\phi]$. As a result of the introduction of new machines $[\theta + \phi, \theta + \phi + d\phi]$, managers may now hire better subordinates while workers are matched with worse managers than before due to greater competition for managers. Thus, technological change is skill-biased in the sense that managers' wages rise but workers' wages generally fall, possibly except for those near the threshold z.[15] The following proposition shows the distributional effects of technology between managers and workers.

Proposition 8 (Skill-biased Technological Change)**.** *Suppose organizations have two layers with managers supervising workers and machines directly. Also, machines are sufficiently complex, satisfying Assumption 7. Then, greater complexity of machines increases managers' wages but reduces workers' wages.*

Proof. See Appendix A.6. □

Proposition 8 follows because machines complement managers but compete with workers. Thus, new machines provide a larger and improved stock of workers from which managers can choose from and increase competition among workers at the same time.

In addition to the distributional effect between skill groups, the gains are heterogeneous among managers. Specifically, the setup, and models of hierarchies more broadly, produces a superstar effect that is reflected in the convex wage functions. The magnitude of this superstar effect depends on the technology, creating heterogeneous effects even among managers.

Proposition 9 (Cascading Effects of Technological Change)**.** *As machines become more complex, wages rise more for more skilled managers.*

[15] Allowing for z to adjust in equilibrium, the marginal workers may earn more by switching into managers.

Proof. See Appendix A.7. □

The intuition is that the introduction of more productive machines increases the superstar effect through the supermodularity in the production function. Supermodularity matters for positive sorting between managers and workers. Since the most skilled managers hire the best machines, they experience the largest gain in their span of control through technological change, which is reflected in their wage increases.

A corollary of Proposition 9 is that technological change increases the income share of top earners.

Corollary 10 (Rising Top Income Shares)**.** *Let $p \in [0, 1]$ indicate the top earners on the interval $[1 - p, 1]$. Then top income shares increase with ϕ for sufficiently small p.*

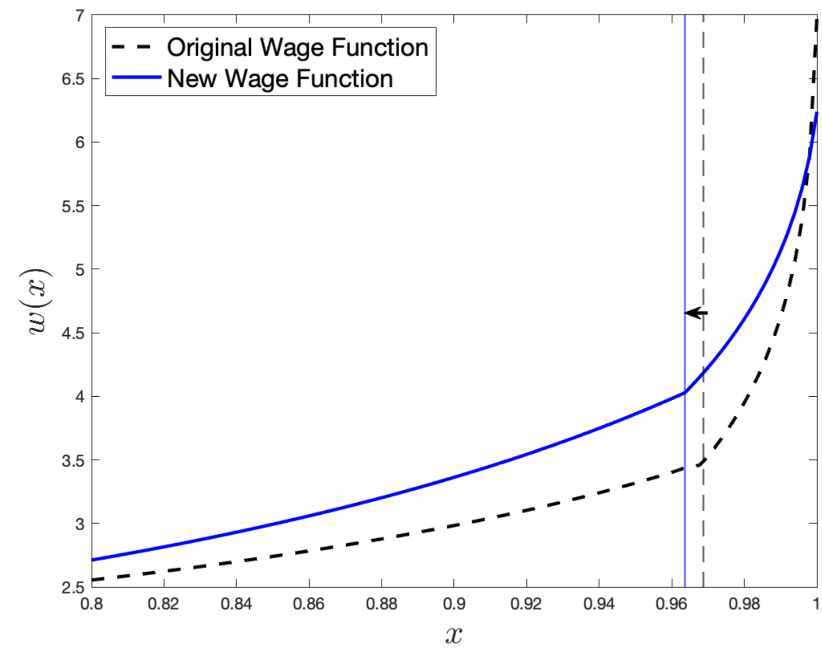
Proof. See Appendix A.8. □

Corollary 10 shows that the model can potentially account for the rise in top labor income shares in the US during the past four decades. This is the period when new technologies, such as personal computers and industrial robots spread across the economy. The result suggests that more advanced technologies favor superstars and lead to greater concentration of income.

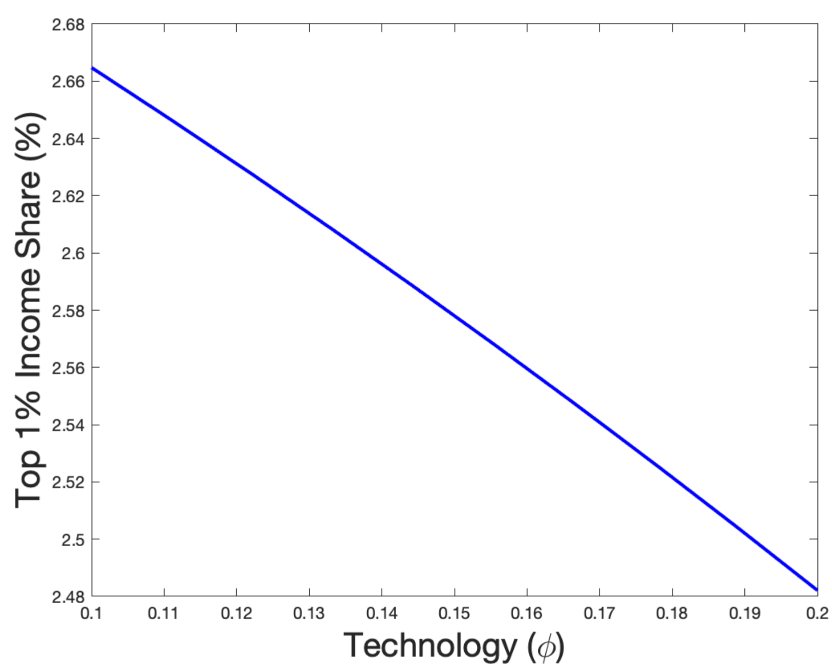
3.2 When Machines Augment Workers

Propositions 8 and 9 extend the results in the literature on skill-biased technological change and automation, which show that skilled (or non-routine) workers benefit more from technological change than unskilled (or routine) workers. Specifically, prop:sbtc establishes a link between gains from technological change and skill levels. As in the existing work, the high-skilled group (that is, managers) gains from technological change while the low-skilled group (that is, workers) loses. Proposition 9 goes beyond the existing results and shows that the gains from technological change are increasing in skill level and thus concentrated at the most skilled agents within the high-skilled group.

Another aspect of the current model that extends the previous work is the role of technology level. As the beginning of this section alludes to, the model may have different implications for top income if machines are not advanced enough to replace workers. As Proposition 6 states,



(a) Changes in wages



(b) Top 1% income share and advances in machines

Figure 7: When machines compete with workers

machines complement workers if the level of technology is low, and it turns out that technological change reduces income concentration at the top.

Proposition 11 (Effects on Top Income). *Technological change reduces top income shares.*

Proof. See Appendix A.9. □

The proof of the result shows that technological change increases workers' wages and thus distributes the output of an organization in favor of the workers. As technological change introduces more productive machines, workers supervise more advanced machines and solve more problems. On the contrary, managers do not directly benefit from technological change because their subordinates remain the same with the occupational threshold fixed. Since the span of control, and thus the skill level of subordinates, is what matters for the equilibrium wages, managers have only limited gains. Moreover, in this setup, technological change is an equalizing force among managers. That is, less skilled managers gain more from technological change than more skilled managers.

Along with Proposition 8, the above result shows that technological change can have opposite effects depending on the level of technology. If machines can only solve the easiest problems and thus are able to assist workers but not managers, workers gain the most from technological change. However, the effect can reverse when technology is sufficiently advanced so that machines become capable of most or all of the tasks assigned to workers.

Figure 7 illustrates the effects of technological change with three layers. Panel (a) shows the changes in the wage function. As the solid curve indicates the most skilled workers and the least skilled managers gain the most. On the other hand, the most skilled managers can experience a fall in their wages. The mechanism behind is that technological change brings the largest productivity gains to the most skilled workers. Thus, workers' wages rise, which more than offsets the gains to managers in the figure. In other words, the distribution of output shifts towards workers from managers. As a result, technological change acts as an equalizing force that reduces income concentration. Top 1% income share in panel (b) is a decreasing function of ϕ, unlike panel (b) of Figure 6.

3.3 Discussion on the Trend in Top Income Inequality and Technology

The model provides a potential explanation for the diverging trend in top income shares around the early 1980s in the U.S. First, note that top income shares were declining during the mid-20th century before increasing abruptly since the 1980s. This was when machines were relatively rudimentary in terms of the tasks they could perform. For example, until the late 1970s industrial robots had limited applications to a relatively narrow range of tasks (Gasparetto, Scalera et al., 2019).

But since the 1980s industrial robots gained more flexibility and became capable of performing significantly more advanced computations than before.[16] The 1980s is also when robots spread across various sectors outside of the automotive industry. In the language of the model of this paper, machines have become comparable to workers in terms of the complexity of problems they can solve as in Section 3.1. One of the main contributions of the model is to generate this non-monotonic relationship between technological change and income concentration.

Relatedly, evidence suggests that technological advances have contributed to the rise of high-income professional workers such as lawyers and investors, allowing them to operate at a greater scale (Kaplan and Rauh, 2010). This is consistent with the results in this section since more complex machines increase the size of the organizations that the most skilled managers supervise.

4 Algorithmic Management and Income Distribution

The analysis so far assumes that machines are only capable of production tasks, that is, producing problems. Specifically, I have intentionally restricted the attention to whether machines replace workers or not. So machines in the previous setup are "narrow" in the sense that they are only suitable for a small set of relatively simple tasks.

However, recent advances in AI have allowed machines to be capable of a broader range of tasks, including those related to management. For example, the latest vision processing technology allows automation of inventory management. The application of AI in recruitment is a relevant example for human resources.

[16] As Gasparetto, Scalera et al. (2019) write, the 1980s was "the time when the robots became even more versatile, by exploiting important improvements both with respect to the hardware and the software."

By extrapolating from these latest advances, one can imagine a scenario where machines become fully capable of running organizations. In this section, I ask: What are the implications of algorithmic management, or "machine managers," for income distribution, especially top incomes? Will future AI systems have qualitatively different effects on income inequality compared to previous automation technologies? I answer these questions by modifying the model so that machines can substitute for managers and have a lower supervision cost than managers.

4.1 Modifications

Denote the supervision cost of machine managers by h_m, which is potentially different from h. In particular, I am interested in the case where $h_m < h$. This implies that machines are more efficient in supervising than managers and thus are capable of forming larger organizations. Nonetheless, machines are still equally efficient in production as workers. In other words, machines have comparative advantage in supervision.

By considering a lower supervision cost of machines, I can examine the effects of technological change on knowledge workers. As in Garicano (2000), the model can be interpreted as follows. Workers specialize in production that requires physical activities. Unlike managers, workers provide mainly the physical resources that are less affected by technologies for cognitive tasks.[17] On the other hand, managers specialize in providing knowledge that complements these physical activities. Thus, technological change in this section can be interpreted as advances in cognitive automation.

4.2 Equilibrium with Algorithmic Management

I proceed as in Section 3.1 and focus on the equilibrium in which organizations have two layers. Agents solve the same problems as before. Machine managers solve

$$w_2(x_2; h_m) = \max_{x_1} \frac{x_2 - w_1(x_1)}{h_m(1 - x_1)}.$$

Note that the wage function $w_2(\cdot)$ of machines is different from that of human managers because of h_m. However, the equilibrium wage function $w_1(\cdot)$ of workers does not directly depend on h_m

[17] For example, even though GPS can perfectly find routes, Uber drivers still need to drive cars to the destination.

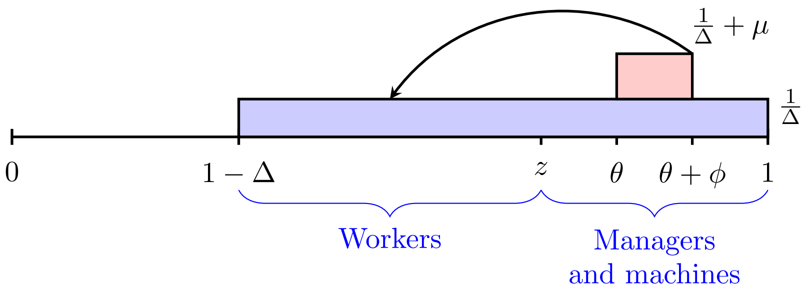


Figure 8: Assignment of machine managers

because the first-order condition is the same as (4). Thus, given a wage schedule, both managers and machines with the same skill level hire the same workers because they have the same first-order conditions.

Note that any dependence of w_1 on h and h_m is through the assignment function. For any value x_1 on $[1 - \Delta, z]$, the assignment equation becomes

$$f(x_1) + g(x_1) = \left[\frac{1}{h(1 - x_1)} \cdot f\left(a(x_1)\right) + \frac{1}{h_m(1 - x_1)} \cdot g(a(x_1)) \right] a'(x_1) \tag{5}$$

As before, the left-hand side is the demand for workers by managers. The right-hand side is the supply of workers. Generally, both the supply and demand are a sum of agents and machines.

If $h_m = h$ then the assignment function takes the standard form. Also, there exists an occupational threshold z that divides workers and managers as before. Machines are equivalent to workers or managers with the same skill level depending on whether they are below or above the threshold.

Figure 8 illustrates how machine managers affect the allocation of workers and managers. Unlike in Figure 4, machines now *reduce* the competition among workers for managers as the density on the interval $[\theta, \theta + \phi]$ increases by the density of machines μ. In other words, machines increase the supply of managers.

To isolate the effects of technological change through algorithmic management, I restrict the technology parameters so that all machines become managers in equilibrium.

Lemma 12 (Machine Managers). *Given the supervision cost h_m of machines, if machines are sufficiently productive then all machines become managers in equilibrium.*

Proof. See Appendix A.10. □

Note that the set of workers is divided into three intervals if $z < \theta$: $[1-\Delta, \underline{y}]$, $[\underline{y}, \bar{y}]$, and $[\bar{y}, z]$ where $1-\Delta < \underline{y} < \bar{y} < z$. The first interval is the set of workers who are matched with the mangers below θ. The second interval is the set of workers who are matched with machines and managers with equivalent skill levels. Lastly, the third interval is the set of workers who are matched with the managers above $\theta + \phi$. Note that the thresholds $\underline{y}$ and $\bar{y}$ depend on parameters θ, ϕ, and h_m.[18] The following lemma summarizes the segregation of workers.

Lemma 13 (Segregation of Workers)**.** *If machines are sufficiently productive then workers are segregated depending on their managers. That is, there exist threshold values* $\underline{y}$ *and* $\bar{y}$*, with* $\underline{y} < \bar{y}$*, such that (i) workers on* $[1-\Delta, \underline{y}]$ *are supervised by managers less productive than machines; (ii) workers on* $[\underline{y}, \bar{y}]$ *are supervised by machines; (iii) workers on* $[\bar{y}, z]$ *are supervised by managers more productive than machines.*

Proof. See Appendix A.11. □

Using the assignment function, I solve for the wage function using the first-order condition of managers. As in Section 2, the equilibrium wage function in this economy is continuous, monotonically increasing, and convex. Notably, machines earn more than managers because $h_m < h$ and they are matched with the same workers paying them the same wages if they have the same skill levels.

4.3 Distributional Effects of Machine Managers

To see the distributional effects of advances in machine managers, consider an increase in ϕ as before. First, advances in machines increase the mass of agents that become workers. To see this, note that new machines allow more workers to be supervised by machines both because there are now more machines in the economy and new machines supervise more workers than the existing ones. The demand for workers rises, leading to increases in workers' wages. As a result, the threshold skill level z rises and the mass of managers decreases.

Lemma 14. *Technological change increases the occupational threshold* z*, and thus reduces the number of managers.*

[18] See Appendix B.4 for the assignment functions.

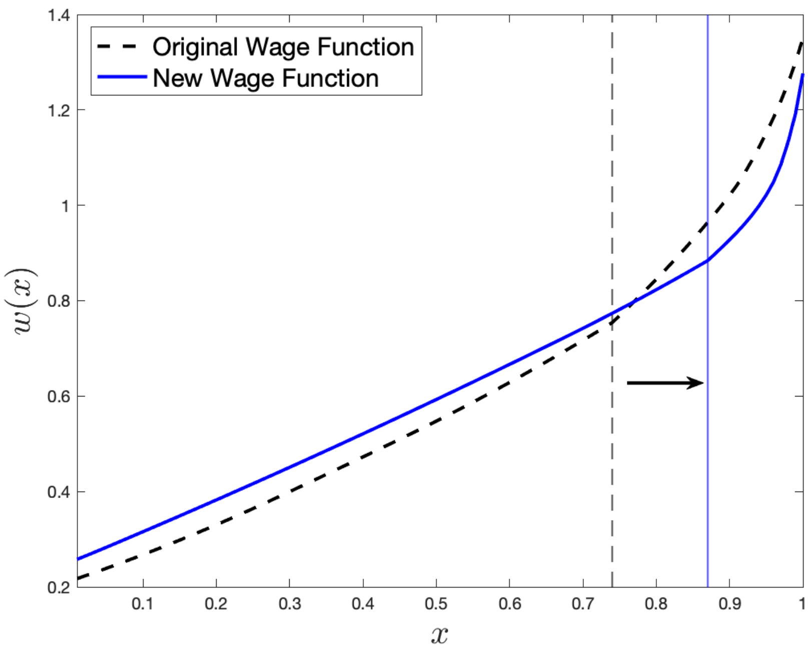


Figure 9: Wage functions when $h_m = h$

Proof. See Appendix A.13. □

An increase in ϕ has two offsetting effects on managers' wages. The first effect is that more advanced machines increase the demand for workers and their wages. Thus, higher wages of workers reduce managers' wages as the total compensation for workers within organizations increases. On the other hand, the second effect is that, as z increases, managers are matched with more skilled workers, and thus they can hire more skilled workers.

Figure 9 shows that the first effect generally dominates the second, and thus most agents who were previously managers (on the right of the dashed vertical line) experience falling wages. The result is somewhat similar to that in Section 3.2. The main difference is how machines augment workers. In Section 3.2, machines augment workers "from below" because they solve easier problems and allow workers to focus on more complex ones. On the other hand, machines in this section augment workers "from above" because they solve more difficult problems than workers. Technological change makes workers more productive because more larger fractions of the problems they generate are solved.

4.4 Supervision Costs and the Nonrivalry of Machines

Machines in the current setup best represent softwares that automate complex managerial tasks. Unlike individual physical machines, AI systems can be deployed to multiple instances simultaneously. In particular, if computations are cheap, then it is possible to deploy AI systems at a large scale without affecting its performance for each instance (e.g. algorithmic management at ride-sharing companies). In an extreme case where computation costs are negligible, the AI systems become almost "nonrivalrous" as the inference costs do not limit the scale at which they are deployed.[19]

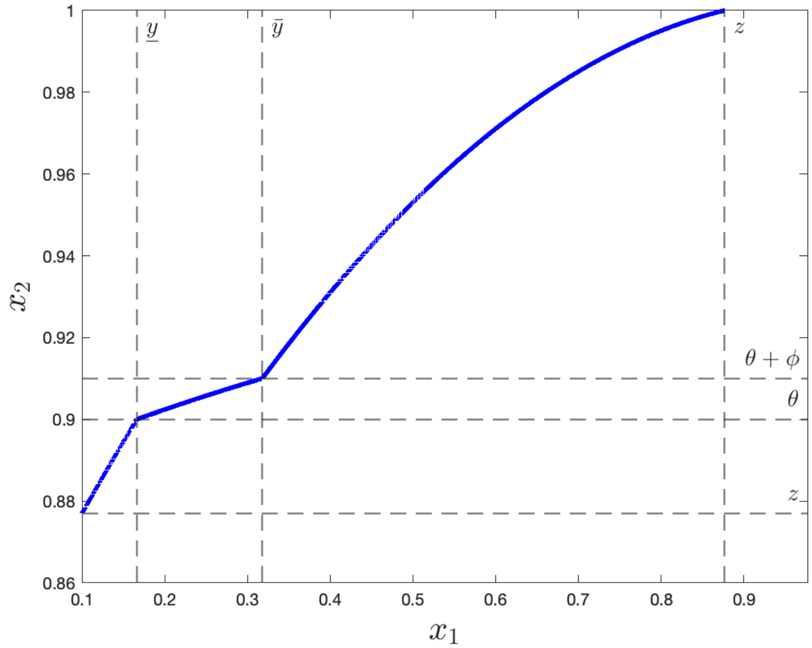
In the current model, h_m represents the cost of computation for such tasks. As a starting point, I consider the case where h_m is slightly lower than h, so that machines are more efficient in supervision than managers but the gap is not too large. In this case, the assignment function has kinks at the ends of the interval of machines as Figure 10a shows.

Figure 10a illustrates the assignment function in the case where $h_m < h$. The x-axis is the skill level of workers on $[1 - \Delta, z]$. The y-axis is the skill level of managers on $[z, 1]$. The dashed vertical lines labeled $\underline{y}$ and $\bar{y}$ divide the workers into three groups as explained above. Notice that the assignment function is flatter on the middle region. Intuitively, workers with different skill levels are supervised by relatively similar managers (or machines). Because there are more managers on $[\theta, \theta + \phi]$ than other points on $[z, 1]$, there is less competition for managers among workers. The flat part of the figure implies that machines with $h_m < h$ allow a larger group of workers to be supervised by the managers on $[\theta, \theta + \phi]$. As h_m falls the middle part of the assignment function becomes even flatter.

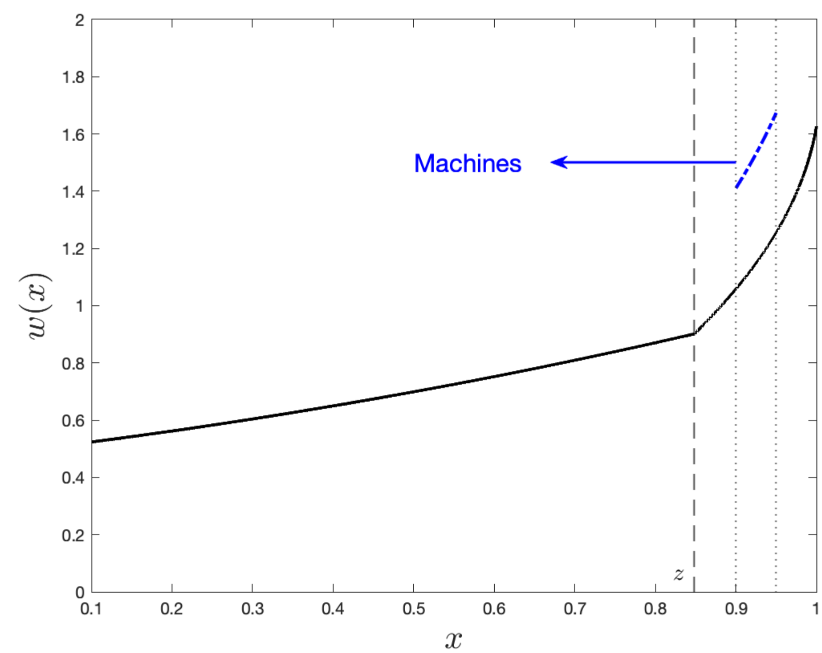
Figure 10b depicts the wage functions. The solid curve is the wages of workers and managers separated by the vertical dahsed line at z on the x-axis. Machines earn wages according to the dash-dotted curve that is located strictly above the solid curve. Intuitively, machines have a lower cost of supervision which allows them to supervise a larger mass of workers than managers with the same skill levels. Therefore, machines earn a higher level of income.

What happens as h_m declines further? Changes to the assignment function are illustrated in Figure 11a. The dotted curve is the assignment function with $h_m = h$, which is the case where

[19] In addition to ride-sharing companies, another example is language models such as ChatGPT and Claude, and their limits on usage due to computational constraints.

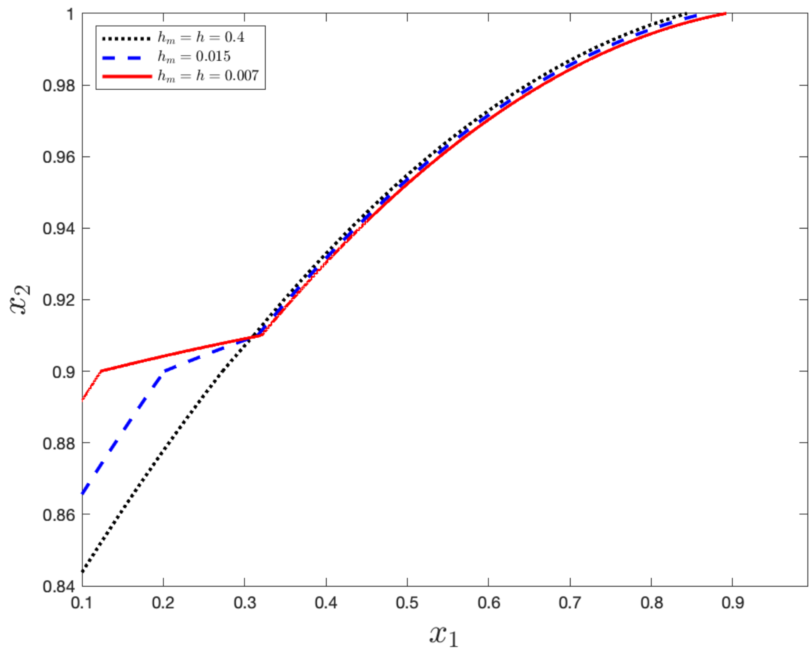


(a) Assignment



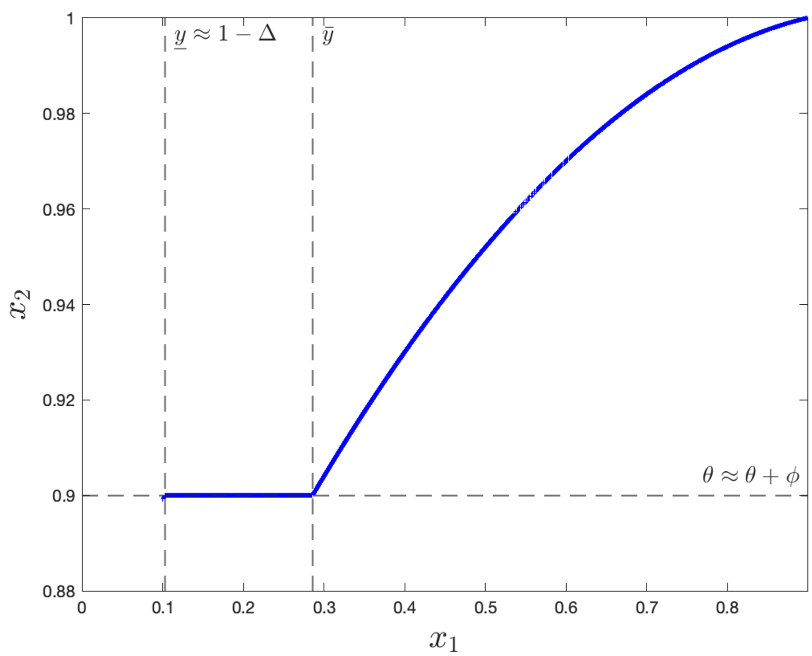
(b) Wages

Figure 10: Assignment and wages when $h_m < h$



(a) Falling h_m and flattening assignment function

Note: The dotted curve is the case where $h_m = h$. As h_m falls, the assignment function changes to the dashed curve and then the solid curve.



(b) Assignment function with nonrivalrous machines

Figure 11: Assignment function and the supervision cost h_m of machines

machines and managers are perfectly substitutable. As h_m falls, the curve on $[\underline{y}, \bar{y}]$ becomes flatter. Moreover, $\underline{y}$ falls as well, meaning that the set of workers supervised by the least skilled managers shrinks.

In the limit where h_m approaches some very low value, workers are segregated into two groups: those supervised by machines (and managers on the overlapping region) and managers who solve more complex problems than machines. In this limit case, all workers below $\bar{y}$ are supervised by machines. Moreover, the mass of workers supervised by machines is not constrained by the time endowment.

Proposition 15 (Nonrivalry of Machines)**.** *Suppose workers are segregated as in Lemma 13. Then the occupational threshold z increases as the supervision cost h_m of machines declines. Moreover, for given values of θ, ϕ, and μ, there exists $\underline{h}$ such that as h_m approaches $\underline{h}$ from above, (i) $z \to \theta$ and (ii) $\underline{y} \to 1 - \Delta$. In other words, for any positive mass of machines, all workers below some threshold are supervised by machines if the supervision cost h_m is sufficiently low.*

Proof. See Appendix A.12. □

A key takeaway from Proposition 15 is that machines become nonrivalrous in the limit case as $h_m \to \underline{h}$. In other words, time is not scarce for machines any more. As a result, workers are segregated into two groups: the more skilled group, who are supervised by managers, and the less skilled group, who are supervised by machines.

Also, the result holds for arbitrarily small values of μ and ϕ. As the proof of Proposition 15 suggests, even if there is a very small mass of machines, because of small values of μ, the segregation result holds for sufficiently small h_m. Thus, in the limit case where $\phi \to 0$, $\mu \to 0$, and $h_m \to 0$, machines are nonrivalrous in the sense that the cost of the supervision of an additional problem is negligible. I interpret this case as a "single" software supervising a large number of workers.

Note that the segregation result applies as long as θ is sufficiently high. Thus, as θ approaches one, it is straightforward to imagine that the productive group shrinks because machines supervise more and more workers. The following result is a corollary of Proposition 15 that verifies this intuition.

Corollary 16 (Nonrivalry and the Automation of Management)**.** *Suppose machines are nonrivalrous in the sense of Proposition 15. Then as θ approaches one all agents become workers.*

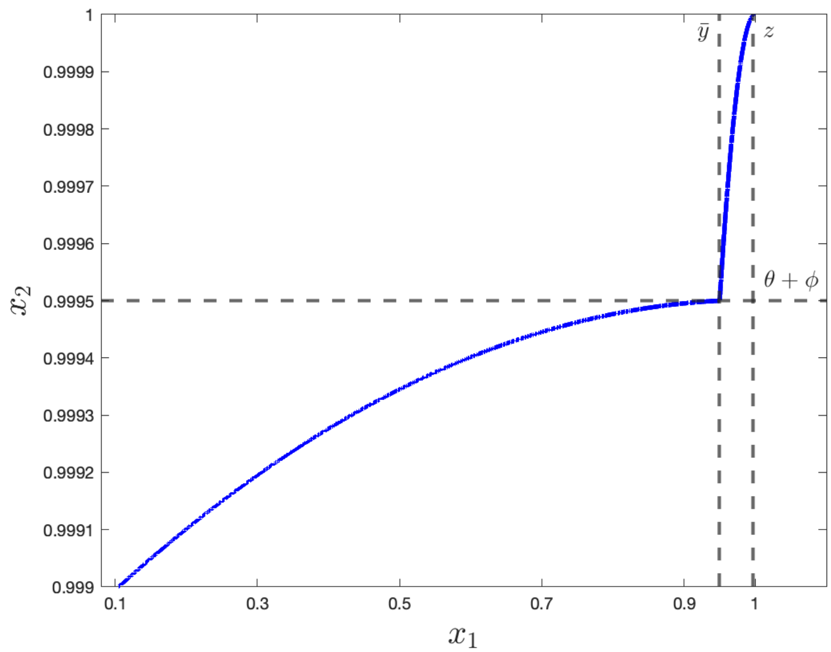


Figure 12: Automation of management

Corollary 16 allows for a speculation about the implications of advanced AI systems on organizational structure, occupational choice, and income distribution. As θ approaches one, machines become comparable to the most skilled managers in their ability to solve problems. As a result, all workers benefit from switching their managers to the machines because of the complementarity between workers and managers. And the nonrivalry of machines implies that all workers are supervised by machines at the limit $h_m \to \underline{h}$.

Figure 12 illustrates how the automation of management affects the assignment function. Notice that, compared with Figure 11b, technology level θ is closer to one (at 0.999) and the heterogeneity among machines is very small ($\phi \approx 0$). Moreover, machines supervise all workers below $\bar{y}$, which is almost all workers in the economy. The remaining managers are those who are more skilled than machines (above $\theta + \phi = 0.9995$) and supervise workers on $[\bar{y}, z]$, which is much smaller than before. According to Corollary 16, as θ approaches one, the steep part of the assignment function on $[\bar{y}, z]$ collapses and the flatter part dominates.

The limit case illustrated in Figure 12 captures a world where all workers are supervised by an algorithm that outperforms any humans.[20] The model predicts that such algorithms have

[20]Such an algorithm would be a superintelligence (Bostrom, 2014).

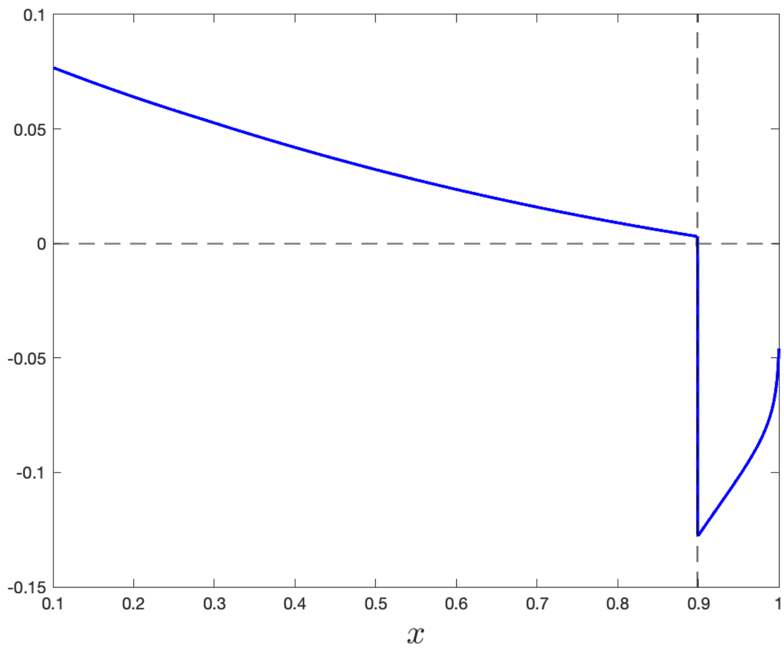


Figure 13: Normalized changes in wages ($\frac{\partial w}{\partial \phi}\frac{\phi}{w}$) as ϕ increases

equalizing effects by erasing productivity differences between workers that arise from managers' skills. In other words, advanced algorithms spread the knowledge to less skilled workers, which only the most skilled workers and managers had. Therefore, the limit case suggests a possibility that a superintelligence significantly reduces labor market inequality by eliminating worker differences due to managerial quality.[21]

Nonrivalrous Machines and Income Inequality Nonrivalrous machines have similar effects on income distribution as in the case with $h_m = h$. Suppose $h_m \approx 0$ so that $\underline{y} \approx 1 - \Delta$ and $z \approx \theta$ as discussed in Proposition 15. Moreover, assume ϕ is small so that machines are on a very narrow interval.

Figure 13 illustrates how wages change at each skill level as machines advance. The vertical dashed line is the threshold z and the blue solid curve is the changes in wages as the parameter ϕ increases by a small amount. As the figure shows, workers' wages rise due to the advances in machines, while managers experience falling wages. Note that it is the rise in the demand for the

[21] However, this result may be relevant only in the short run as a software superintelligence is developed but constrained by physical actuators.

relatively less productive workers on $[\underline{y}, \bar{y}]$ that drives the overall upward shift in workers' wage function. Intuitively, increases in wages on $[\underline{y}, \bar{y}]$ lead to increases in wages on the other parts on $[1-\Delta, z]$ due to the monotonicity of the wage function. Thus, the gains from technological change trickles up from less skilled to more skilled workers.

In addition to falling income inequality between workers and managers, it turns out that technological change has different distributional implications among managers compared to workers. In particular, wages decline more for less skilled managers than more skilled managers. As a result, income inequality *among* managers rises. Intuitively, technological change has a first-order negative effect on managers' wages by increasing the compensation for workers. At the same time, managers face a larger supply of workers through the increase in z, which tends to offset the first effect. However, the most skilled managers are those who benefit from the second effect. More skilled managers gain increasingly more from this second effect. Thus, the equilibrium wage function $w_2(\cdot)$ becomes more convex.

It is worth noting that the results on falling income inequality among workers is consistent with early evidence on the effects of AI. Brynjolfsson, Li, and Raymond (2023) find that low-skilled workers gain more from AI in the context of call centers. Their interpretation is that AI spreads knowledge to low-skilled workers who have less experiences than high-skill workers and lack such knowledge. Noy and Zhang (2023) and Peng et al. (2023) find similar results in other settings and report equalizing effects of AI. Through the lens of the model, AI managers help workers by sending them the knowledge required for solving problems. More skilled workers do not gain as much because they need supervision for problems that AI managers cannot solve either.

5 Conclusion

The paper was partly motivated by the discussion on who will be augmented by AI. While AI could be mainly a continuation of previous automation technologies examined in the literature, early evidence suggests the opposite may happen. The results of the paper show that a crucial factor is the maximum complexity of automated tasks. In particular, the model highlights the vertical structure of production processes and how AI would fit into it.

On the concerns related to AI automation and income inequality, the model predicts the divergence in income distribution observed over the past four decades may persist. As AI workers grow increasingly useful, human workers whose skills have become more abundant will face a greater competition. On the other hand, those who can leverage their skills over other workers, either human or AI, are more likely to succeed.[22] The difference from the previous waves of automation is that the degree of income concentration can be greater.

The model also suggests a possibility that future AI systems reduce income concentration as they substitute for high-skilled workers. Thus, compared to AI workers, AI managers can have a positive effect on worker demand. It is possible that both AI workers and AI managers affect the labor market as technology advances. The net effect on income inequality depends on which of these forces dominates.

There are several avenues for future work. First, an unexplored channel in this paper is the implications of technology ownership for income and wealth inequality. As the results in Section 4 suggest, owners of technology can earn significantly higher shares of income as technology advances. As workers earn smaller shares of total income, the ownership of technology may become a major determinant of overall income inequality. Second, recent progress in AI presents potential for artificial general intelligence (AGI), or complete automation of tasks done by humans, which can have profound implications for economic growth and overall labor demand. Korinek and Suh (2023) is one attempt to examine various possibilities brought by AGI. Lastly, it is important to understand the welfare effects of technological change and policy implications, which can be challenging because of the presence of market imperfections and limited policy tools.

[22] In a related context, "agency" may be particularly important as AI advances (Seetharaman and Wells, 2023).

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A Proofs

A.1 Proof of Lemma 2

Suppose a manager x_2 supervises workers x_1 and $x_1' < x_1$. There can be workers on (x_1', x_1) who are supervised by managers less skilled than x_2. The planner can reallocate the time of the manager x_2 away from workers x_1' to more skilled workers on (x_1', x_1) by replacing the less skilled managers. This is because of the supermodularity of the production function. Likewise, there can also be workers on (x_1', x_1) who are supervised by managers more skilled than x_2. Then the planner can increase output by reallocating the time of the manager x_2 away from workers x_1 to the less skilled workers on (x_1', x_1).

Also, note that managers of a single skill level cannot supervise a continuum of workers with a positive measure because agents at each skill level has zero measure. Otherwise, the labor market clearing condition is violated, which requires that the supply of workers of a positive measure is met with the same measure of demand from managers. See Antras, Garicano, and Rossi-Hansberg (2005) for related discussions.

A.2 Proof of Lemma 3

Suppose that the set of human workers is $[1-\Delta, y_1]\cup[y_2, y_3]$ and the set of managers is $[y_1, y_2]\cup[y_3, 1]$ with $1 - \Delta < y_1 < y_2 < y_3 < 1$ so the set of workers is not connected. For this allocation to be an equilibrium, the wage functions must be continuous at the thresholds y_1, y_2, and y_3. Otherwise, marginal agents have the incentive to switch into occupations that yield higher income. Moreover, agents should not have an incentive to changes matches.

Let $\hat{w}_1(\cdot)$ and $\hat{w}_2(\cdot)$ denote the wage functions of the workers and managers on $[1 - \Delta, y_1]$ and $[y_1, y_2]$. Also, let $\tilde{w}_1(\cdot)$ and $\tilde{w}_2(\cdot)$ denote the wage functions of workers and managers on $[y_2, y_3]$ and $[y_3, 1]$. Now consider the incentive of manager $x_2 = y_3$ who hires workers $x_1 = y_2 - \epsilon$

$$\pi(y_3, y_2 - \epsilon) = \frac{y_3 - \hat{w}_2(y_2 - \epsilon)}{h(1 - y_2 + \epsilon)}$$

$$\frac{\partial \pi(y_3, y_2 - \epsilon)}{\partial \epsilon} = -\frac{\hat{w}_2'(y_2 - \epsilon)}{h(1 - y_2 + \epsilon)} + \frac{y_3 - \hat{w}_2(y_2 - \epsilon)}{h(1 - y_2 + \epsilon)^2}$$

Then the manager does not have an incentive to hire workers $y_2 - \epsilon$ if

$$\left.\frac{\partial \pi(y_3, y_2 - \epsilon)}{\partial \epsilon}\right|_{\epsilon \to 0} = -\frac{\hat{w}_2'(y_2)}{h(1-y_2)} + \frac{y_3 - \hat{w}_2(y_2)}{h(1-y_2)^2} < 0$$

However, the derivative is in fact positive since

$$-\frac{\hat{w}_2'(y_2)}{h(1-y_2)} + \frac{y_3 - \hat{w}_2(y_2)}{h(1-y_2)^2} = -\frac{\hat{w}_2'(y_2) - \tilde{w}_1'(y_2)}{h(1-y_2)}$$

and

$$\hat{w}_2'(y_2) = \frac{1}{h(1-y_1)} > 1 > \frac{y_3 - \hat{w}_2(y_2)}{1-y_2} = \tilde{w}_1'(y_2)$$

The last inequality follows from $y_3 < 1$ and $\hat{w}_2(y_2) \geq y_2$, which is necessary for all agents to be matched. This shows that an allocation where the set of workers is disconnected cannot be an equilibrium.

A.3 Proof of Lemma 4

Wages increase in skill level. Given a wage schedule, since machine owners maximize income, they enter only if their wages are greater than the entry cost. Higher μ makes the assignment function steeper and thus the least skilled machines earn less. If μ is sufficiently high then wages of the least skilled machines may be lower than ϵ, which implies $\theta^* \geq \theta$.

A.4 Proof of Proposition 5

For each statement in the proposition, I start by showing that there exists a unique allocation satisfying the optimality conditions of managers and the labor market clearing conditions. Then I argue that such an allocation is an actual equilibrium by showing that agents have no incentive to deviate from the given allocation.

Existence and uniqueness of a solution: An equilibrium is characterized by the two differential equations for the optimality of the manager's problem and the market clearing condition. Since the coefficient functions are continuous, these differential equations have a unique solution (Boyce and DiPrima, 2020).

Equilibrium: The next step is to show that the allocation is actually an equilibrium. To begin with, note that h must not be too low for the economy to have an equilibrium. Given μ, h must be above some threshold to ensure $z < 1$. Intuitively, if there is a large amount of machines and managers are very efficient (low h) then the supply of problems (or workers and machines) is not met by the demand from managers.

In the following, I show that agents do not have the incentive to deviate when fixing the number of layers at two. As Lemma 3 shows, the set of workers is connected and thus the allocation is characterized by the threshold z. First, agents slightly below z do not become managers. Suppose workers slightly below z by $\delta > 0$ deviate by hiring the least skilled workers ($x_1 = \theta$). Then they earn less wages. To see this, consider the following derivative

$$\frac{\partial w_1(z-\delta)}{\partial \delta} = -w_1(z-\delta)$$

By setting $\delta = 0$ I get

$$\left.\frac{\partial w_1(z-\delta)}{\partial \delta}\right|_{\delta=0} = -w_1(z) < 0$$

If these workers hire the least skilled workers θ as managers then the payoff would be

$$\frac{z-\delta-w_1(\theta)}{h(1-\theta)} = w_2(z) - \frac{\delta}{h(1-\theta)}$$

Taking the derivative of the deviation payoff, I get

$$\frac{\partial}{\partial \delta}\frac{z-\delta-w_1(\theta)}{h(1-\theta)} = -\frac{1}{h(1-\theta)} < 0$$

Note that $w_1'(z) = \frac{\hat{x}_2 - w_1(z)}{1-z}$. Also, $\hat{x}_2 < 1$ under Assumption 7 and $w_1(z) > z$ since I am interested in an equilibrium where all agents are matched. Thus, $-w_1'(z) > -1/h(1-\theta)$, which implies workers slightly below z are worse off if they become managers.

Agents slightly above z do not have an incentive to deviate by switching occupations. Because there are machines and managers' wages increase more steeply at z, the most skilled managers do not hire agents slightly above z as workers.

Now I show that the managers do not have the incentive to add additional layers by examining the incentive of the most skilled managers $x_2 = 1$. Suppose that a manager with skill one deviates and switches to an organization with three layers. Since machines cannot supervise, the manager hires workers $x_1 \in [1 - \Delta, z]$ as his direct subordinates. Also, workers $x_0 < x_1$ are hired in the bottom layer. The manager pays the prevailing market wages $w_1(x_1)$ and $w_1(x_0)$ to the wprkers. The manager, thus, solves

$$\hat{w}_2(x_2) = \max_{x_0, x_1} \hat{n}_1 \hat{n}_0 x_2 - \hat{n}_1 \hat{w}_1(x_1) - \hat{n}_1 \hat{n}_0 \hat{w}_1(x_0)$$

subject to

$$h(1 - x_1)\hat{n}_1 \hat{n}_0 \leq 1$$
$$h(1 - x_0)\hat{n}_0 \leq 1$$

where the hat indicates the variables associated with the deviating manager. Note that $\hat{w}_1(\cdot)$ is the workers' wages which equal $w_2(x)$ if $x \geq z$ and $w_1(x)$ if $x < z$. The manager has no incentive to pay strictly above the equilibrium wages because workers will accept any offer greater than the market wage. Also, the manager cannot pay below because the workers will not accept the offer.

Substituting the time constraints into the objective function and taking the derivatives, I obtain

$$[x_0] \quad \hat{w}_1(x_1) = \frac{1}{h}\hat{w}_1'(x_0)$$
$$[x_1] \quad \frac{x_2 - \hat{w}_1(x_0)}{h(1 - x_1)^2} = \frac{1 - x_0}{(1 - x_1)^2}\hat{w}_1'(x_1) + \frac{1 - x_0}{1 - x_1}\hat{w}_1(x_1)$$

Note that the optimality of the deviating manager requires $x_1 \geq x_0$, otherwise the manager may benefit by directly supervising the workers in the bottom layer. Then there are three cases to consider: (i) $x_0 \leq \theta + \phi$, $x_1 \geq z$, (ii) $x_0 < x_1 \leq z$, and (iii) $\theta + \phi \leq x_0 < x_1$.

Consider the first case. The FOCs are

$$
\begin{aligned}
&[x_0] \quad w_2(x_1) = \frac{1}{h} w_1'(x_0) \\
&[x_1] \quad \frac{x_2 - w_1(x_0)}{h(1-x_1)^2} = \frac{1-x_0}{(1-x_1)^2} w_2(x_1) + \frac{1-x_0}{1-x_1} w_2'(x_1)
\end{aligned}
$$

The FOC with respect to x_0 can be rewritten as

$$
\frac{x_1 - w_1(a^{-1}(x_1))}{h(1-a^{-1}(x_1))} = \frac{1}{h} \frac{a(x_0) - w_1(x_0)}{1-x_0}
$$

where the left-hand side uses the equilibrium wage function of manager and the assignment function. The right-hand side uses the first-order condition of managers in the original allocation and the assignment function. Note that the equation is $w_2(x_1) = w_2(a(x_0))$. Since the equilibrium wage function $w_2(\cdot)$ is monotonically increasing in x_1, it is one-to-one. Thus, the pair (x_0, x_1) must satisfy

$$
x_1 = a(x_0)
$$

With three layers, the manager may reduce the amount of compensation paid to workers by hiring more productive machines that allow the manager to hire fewer workers. Together with the above relationship, the FOC with respect to x_1 pins down the optimal choice of x_0 and x_1. Rearrange the terms to obtain

$$
\frac{x_2 - w_1(x_0)}{h(1-x_0)} = w_2(x_1) + (1-x_1) w_2'(x_1)
$$

Using the conditions that characterize the original allocation, the right-hand side can be written as

$$
\begin{aligned}
w_2(x_1) + (1-x_1) w_2'(x_1) =& \frac{x_1 - w_1(a^{-1}(x_1))}{h(1-a^{-1}(x_1))} + (1-x_1) \frac{1}{h(1-a^{-1}(x_1))} \\
=& \frac{1 - w_1(a^{-1}(x_1))}{h(1-a^{-1}(x_1))}
\end{aligned}
$$

Together with $x_1 = a(x_0)$, I have

$$\frac{x_2 - w_1(x_0)}{h(1-x_0)} = \frac{1 - w_1(x_0)}{h(1-x_0)}$$
$$\implies x_2 = 1$$

which is true only if $x_2 = 1$ in the first place. In other words, if $x_2 = 1$ any choice of subordinates satisfying $x_1 = a(x_0)$ yields the same payoff. Note that if $x_2 < 1$ then the deviating manager chooses $x_0 = \theta$ and $x_1 = z$ to avoid paying high labor costs from hiring more productive subordinates. The payoff of manager x_2 is then

$$\hat{w}_2(x_2) = \frac{x_2}{h(1-a(x_0))} - \frac{1-x_0}{1-a(x_0)} w_2(a(x_0)) - \frac{w_1(x_0)}{h(1-a(x_0))}$$

Substituting in $w_2(a(x_0)) = \frac{a(x_0) - w_1(x_0)}{h(1-x_0)}$, I obtain

$$\begin{aligned}
\hat{w}_2(x_2) &= \frac{x_2 - w_1(x_0) - h(1-x_0)w_2(a(x_0))}{h(1-a(x_0))} \\
&= \frac{x_2 - w_1(x_0) - a(x_0) + w_1(x_0)}{h(1-a(x_0))} \\
&= \frac{x_2 - a(x_0)}{h(1-a(x_0))}
\end{aligned}$$

For $x_2 = 1$, it follows that $\hat{w}_2(x_2) = 1/h$. Note that $w_2(1) = \frac{1-w_1(\theta+\phi)}{h(1-(\theta+\phi))}$. Thus, the manager does not have an incentive to deviate if $\frac{1-w_1(\theta+\phi)}{h(1-(\theta+\phi))} > \frac{1}{h}$, which is equivalent to

$$w_1(\theta + \phi) < \theta + \phi \tag{6}$$

Condition (6) is true if the supply of machines μ at each point is sufficiently large so that workers' wages are low.

Now I turn to the second case where $x_1 \leq z$. In this case, the first-order conditions are

$$\begin{aligned}
&[x_0] \quad w_1(x_1) = \frac{1}{h} w_1'(x_0) \\
&[x_1] \quad \frac{x_2 - w_1(x_0)}{h(1-x_1)^2} = \frac{1-x_0}{(1-x_1)^2} w_1'(x_1) + \frac{1-x_0}{1-x_1} w_1(x_1)
\end{aligned}$$

Rearrange the FOC with respect to x_1 to obtain

$$\frac{x_2 - w_1(x_0)}{h(1-x_0)} = a(x_1)$$

And combined with the optimality condition in the original allocation, the FOC with respect to x_0 is

$$w_2(a(x_0)) = w_1(x_1)$$

Since $x_1 \leq z$, the only values of x_0 and x_1 satisfying the above condition are $x_0 = \theta < 1 - \Delta$ and $x_1 = z$. However, it does not generally satisfy the first-order condition with respect to x_1. Thus, there is no solution corresponding to the second case.

Lastly, consider the third case where $x_0 \geq \theta + \phi$. The first-order conditions are

$$\begin{aligned} &[x_0] \quad w_2(x_1) = \frac{1}{h} w_2'(x_0) \\ &[x_1] \quad \frac{x_2 - w_2(x_0)}{h(1-x_1)^2} = \frac{1-x_0}{(1-x_1)^2} w_2'(x_1) + \frac{1-x_0}{1-x_1} w_2(x_1) \end{aligned}$$

First, note that any choice with $x_0 > \theta + \phi$ is not profitable because there is a discontinuous increase from $w_1(\theta+\phi)$ to $w_2(\theta+\phi+\delta)$ for any positive δ. Instead, consider the case where $x_0 = \theta + \phi$ and $x_1 > x_0$. Then the manager $x_2 = 1$ does not have an incentive to deviate if

$$w_2(x_2) = \frac{x_2 - w_1(\theta+\phi)}{h(1-(\theta+\phi))} > \frac{x_2 - w_1(\theta+\phi) - h(1-(\theta+\phi))w_2(x_1)}{h(1-x_1)} = \hat{w}_2(x_2)$$

Note that the deviating manager pays direct subordinates $w_2(x_1)$ since $x_1 > \theta + \phi$. Rearrange the terms on the right-hand side so that

$$w_2(x_1) \frac{1-(\theta+\phi)}{x_1 - (\theta+\phi)} > w_2(1)$$

Note that the left-hand side diverges to infinity as $x_1 \to \theta + \phi$ and approaches $w_2(1)$ as $x_1 \to 1$. A sufficient condition for the inequality to hold is that the left-hand side term is monotonically decreasing in x_1. Take the derivative

$$\frac{dLHS}{dx_1} = \left(1 - \frac{x_1 - w_1(a^{-1}(x_1))}{x_1 - (\theta + \phi)}\right) \frac{1 - (\theta + \phi)}{x_1 - (\theta + \phi)} \frac{1}{h(1 - a^{-1}(x_1))}$$

The derivative is negative if $w_1(a^{-1}(x_1)) < \theta + \phi$, which is true if μ is sufficiently large and thus (6) holds. Thus, managers do not have an incentive to deviate and add another layer in the third case either.

A.5 Proof of Proposition 6

As in Proposition 5, I start by showing that there exists a unique allocation satisfying the optimality conditions of managers and the labor market clearing conditions. Then I argue that such an allocation is an actual equilibrium by showing that agents have no incentive to deviate from the given allocation.

Existence and uniqueness of a solution: Given the assignment function, the equilibrium wage functions w_0 and w_1 solve the system of equations given by the first-order conditions. Rearrange the terms so that

$$[x_0] \quad w_1(a(x_0)) = \frac{1}{h} w_0'(x_0), \quad x_0 \in [\theta, \theta + \phi]$$

$$[x_1] \quad \frac{a(x_1)}{h(1 - x_1)(1 - a^{-1}(x_1))} = w_1'(x_1) + \frac{w_1(x_1)}{1 - x_1} + \frac{w_0(a^{-1}(x_1))}{h(1 - x_1)(1 - a^{-1}(x_1))}, \quad x_1 \in [1 - \Delta, z]$$

Again, the above system of equations has at most a unique solution (Boyce and DiPrima, 2020). Thus, there exists a unique allocation that satisfies the first-order conditions of the managers, the boundary conditions for the assignment function, and the continuity of the wage functions.

Equilibrium: The allocation described above requires θ^* to adjust and clear the labor market. Since the supply of machines on the market cannot be greater than the endowment of machines in the economy, it must be that $\theta^* \geq \theta$ in equilibrium. The demand for machines increases as h falls and thus h must not be too low.

To prove that the allocation described above is an actual equilibrium, I consider whether agents have the incentive to deviate from the allocation described above when (i) the number of layers is fixed and (ii) managers are allowed to choose two layers. Fixing the number of layers at three, I show that machines and workers are in separate layers, and the occupational threshold z separates workers and managers. Since machines cannot supervise others, including other machines, it suffices to show that all workers are in the middle layer.

Consider the incentive of the most skilled manager $x_2 = 1$. Suppose the manager deviates and hires the least skilled workers $x_1 = 1 - \Delta$ in the bottom layer, instead of machines $x_0 = \theta + \phi$. The deviation is not profitable if

$$\begin{aligned} w_2(1) &> \hat{w}_2(1) \\ \frac{w_1(1-\Delta) - w_0(\theta+\phi)}{(1-\Delta) - (\theta+\phi)} &> w_1(z) \end{aligned}$$

Note that the left-hand side is increasing in θ because $w_0(\cdot)$ is convex. On the other hand, the right-hand side is falling in θ because $w_1(z) = w_0'(\theta+\phi)/h$ from the FOC with respect to x_0. Thus, θ must be sufficiently small.

To show that z separates managers and workers, I show that $w_1'(z) < w_2'(z)$. If it were true then the most skilled manager cannot profitably deviate by hiring agents above z. To see this, consider the incentive of the best manager hiring agents slightly above z

$$\pi(1, z+\delta) = \frac{1 - w_0(\theta+\phi)}{h(1-z-\delta)} - \frac{1-\theta-\phi}{1-z-\epsilon} w_2(z+\delta)$$

For the allocation to be an equilibrium, it must be that

$$\lim_{\delta \to 0} \frac{\partial \pi}{\partial \delta} = \frac{1-\theta-\phi}{1-z} \big(w_1'(z) - w_2'(z) \big) < 0$$

which holds if $w_1'(z) - w_2'(z) < 0$.

To find the conditions for the inequality to hold, consider the FOC with respect to x_1 at $x_1 = z$:

$$w_1'(z) = \frac{1}{1-\theta-\phi} \frac{1}{h(1-z)} \left(1 - w_1(z)h(1-\theta-\phi) - w_0(\theta+\phi)\right)$$

Since $w_2'(z) = 1/h\Delta$, I need to show

$$w_1'(z) = \frac{1}{1-\theta-\phi}\frac{1}{h(1-z)}\left(1 - w_1(z)h(1-\theta-\phi) - w_0(\theta+\phi)\right) < \frac{1}{h\Delta}$$

Rearrange the terms to obtain

$$\frac{1}{1-z}\left(\frac{\Delta(1-w_0(\theta+\phi))}{1-\theta-\phi} - w_1(z)h\Delta\right) < 1$$

Since the indifference condition implies $w_1(z) = w_2(z)$, I have

$$w_1(z) = w_2(z) = \frac{z}{h\Delta} - \frac{1-\theta^*}{\Delta}w_1(1-\Delta) - \frac{w_0(\theta^*)}{h\Delta}$$

and substituting the above into the brackets it follows that

$$\begin{aligned}&\frac{\Delta(1-w_0(\theta+\phi))}{1-\theta-\phi} - \left(\frac{z}{h\Delta} - \frac{1-\theta^*}{\Delta}w_1(1-\Delta) - \frac{w_0(\theta^*)}{h\Delta}\right)h\Delta \\ =&\frac{\Delta(1-w_0(\theta+\phi))}{1-\theta-\phi} - z + h(1-\theta^*)w_1(1-\Delta) + \epsilon\end{aligned}$$

I need to show that the last expression is strictly less than $1-z$

$$\frac{\Delta(1-w_0(\theta+\phi))}{1-\theta-\phi} + h(1-\theta^*)w_1(1-\Delta) + \epsilon < 1$$

Rearrange the first term so that

$$\frac{\Delta}{1-\theta-\phi}(1 - w_0(\theta+\phi) - h(1-\theta-\phi)w_1(z)) + h\Delta w_1(z) + h(1-\theta^*)w_1(1-\Delta) + \epsilon < 1$$

Note that $h\Delta w_1(z) = h\Delta w_2(z) = z - w_0(\theta^*) - h(1-\theta^*)w_1(1-\Delta)$ is the wage of manager $x_2 = z$ per machine and thus

$$\frac{1-w_0(\theta+\phi)}{h(1-\theta-\phi)} < \frac{1-w_0(\theta^*) - h(1-\theta^*)w_1(1-\Delta)}{h\Delta} \tag{7}$$

The left-hand side is the payoff of manager $x_2 = 1$ if he deviates and hires only one layer of machines $x_0 = \theta+\phi$. The right-hand side is the payoff of the same manager if he deviates and hires workers $x_1 = 1-\Delta$ and machines $x_0 = \theta$. The condition requires that the first deviation strategy is dominated by the second. This is true for values of θ that are sufficiently small. Suppose θ is large so that θ^* is close to $1-\Delta$ and ϕ is small so that $\theta+\phi \approx \theta^*$. Then the difference between the left-hand side and the right-hand side is

$$\begin{aligned}&\frac{1-w_0(\theta+\phi)}{h(1-\theta-\phi)} - \frac{1-w_0(\theta^*)-h(1-\theta^*)w_1(1-\Delta)}{h\Delta} \\ &\approx w_1(1-\Delta) > 0\end{aligned}$$

which violates (7). If θ^* is sufficiently low then z characterizes the equilibrium allocation of agents.

The next step is to prove that managers do not have an incentive to switch to two layers. I show this in the case of the marginal managers at $x_2 = z$. This is sufficient because more skilled managers only run at least as many layers as less productive managers.[23]

Suppose a manager $x_2 = z$ deviates from the allocation described above and switches to two layers. The deviating manager faces a trade-off between a fall in total labor cost due to fewer layers and a rise in total labor cost due to more direct subordinates. Note that it is more profitable to choose $x_1 \geq 1-\Delta$ than $x_1 \in [\theta, \theta+\phi]$ if $\theta+\phi$ is sufficiently small.[24] Since optimality requires $x_1 < z$, consider the following problem

$$\hat{w}_2(z) = \max_{x_1 \in [1-\Delta, z)} \pi(z, x_1)$$

where $\pi(z, x_1) \equiv \frac{z-w_1(x_1)}{h(1-x_1)}$. First, I show that $x_1 = 1-\Delta$ is dominated by the original choice of organization structure that yields $w_2(z)$ if $(1-h)(1-\Delta)$ is sufficiently large. To see this, consider the difference in payoffs

$$w_2(z) - \pi(z, 1-\Delta) = \frac{(1-h(1-\theta^*))w_1(1-\Delta) - w_0(\theta^*)}{h\Delta} \tag{8}$$

[23] See the discussion on the equilibrium number of layers in Garicano and Rossi-Hansberg (2006)

[24] On the other hand, if $\theta+\phi$ were close to $1-\Delta$ then the manager hires machines as direct subordinates to exploit the fact that $w_0(\theta+\phi)$ is strictly less than $w_1(1-\Delta)$, which is true for sufficiently small ϵ.

which is nonnegative (and so there is no incentive to deviate) if

$$(1-h(1-\theta^*))w_1(1-\Delta) \geq \epsilon$$

The left-hand side is bounded below by

$$(1-h(1-\theta^*))w_1(1-\Delta) \geq (1-h+h\theta) \cdot \frac{1-\Delta-w_0(\theta^*)}{h(1-\theta^*)}$$

where, in the last inequality, $\frac{1-\Delta-w_0(\theta^*)}{h(1-\theta^*)}$ is the outside option of the least skilled workers as a manager supervising machines. In equilibrium, it must be that $w_1(1-\Delta) \geq \frac{1-\Delta-w_0(\theta^*)}{h(1-\theta^*)}$ to rule out deviation of workers $x_1 = 1-\Delta$. The right-hand side is bounded further below by

$$(1-h+h\theta) \cdot \frac{1-\Delta-w_0(\theta^*)}{h(1-\theta^*)} \geq \left(\frac{1}{h(1-\theta)} - 1\right)(1-\Delta-\epsilon)$$

A sufficient condition to (8) is

$$\theta \geq \frac{\epsilon-(1-h)(1-\Delta)}{h(1-\Delta)}$$

The last inequality requires that, given Δ and θ, h must be sufficiently small. Also, the condition trivially holds if $\epsilon-(1-h)(1-\Delta) < 0$ because $\theta \geq 0$.[25]

Now I show that any $x_1 > 1-\Delta$ yields a lower payoff than $x_1 = 1-\Delta$. That is,

$$\frac{z-w_1(1-\Delta)}{h\Delta} \geq \frac{z-w_1(x_1^*)}{h(1-x_1^*)}$$

for $x_1 \in (1-\Delta, z)$. Consider the change in payoff as x_1 rises from $1-\Delta$ to $1-\Delta+\delta$

$$\pi(z, 1-\Delta+\delta) = \frac{z-w_1(1-\Delta+\delta)}{h(1-(1-\Delta+\delta))} = \frac{z-w_1(1-\Delta+\delta)}{h(\Delta-\delta)}$$
$$\frac{\partial\pi}{\partial\delta} = -\frac{w_1'(1-\Delta+\delta)}{h(\Delta-\delta)} + \frac{z-w_1(1-\Delta+\delta)}{h(\Delta-\delta)^2}$$

[25]And I assume that the entry cost ϵ is small.

Letting $\delta \to 0$, I have

$$\left.\frac{\partial \pi}{\partial \delta}\right|_{\delta \to 0} = -\frac{1}{h\Delta}\left(w_1'(1-\Delta) + \frac{w_1(1-\Delta)}{\Delta}\right) + \frac{z}{h\Delta^2}$$

I want to show that $\left.\frac{\partial \pi}{\partial \delta}\right|_{\delta \to 0} < 0$. Note that the FOC with respect to x_1 implies

$$w_1'(1-\Delta) + \frac{w_1(1-\Delta)}{\Delta} = \frac{1}{1-\theta^*}\frac{z-\epsilon}{h\Delta}$$

Substituting the above expression into the derivative, I can write

$$\left.\frac{\partial \pi}{\partial \delta}\right|_{\delta \to 0} = \left(1 - \frac{1}{h(1-\theta^*)}\right)\frac{z-\epsilon}{h\Delta^2} + \frac{\epsilon}{h\Delta^2}$$

The goal is to show that the derivative is negative. Rearrange the terms to obtain

$$\epsilon < \left(\frac{1}{h(1-\theta^*)} - 1\right)(z-\epsilon)$$

Note that the right-hand side is bounded below by

$$\left(\frac{1}{h(1-\theta^*)} - 1\right)(z-\epsilon) > \left(\frac{1}{h(1-\theta)} - 1\right)(1-\Delta-\epsilon)$$

Thus, a sufficient condition for the inequality to hold is

$$\left(\frac{1}{h(1-\theta)} - 1\right)(1-\Delta-\epsilon) > \epsilon$$

Again rearrange the terms so that

$$1 - \frac{1}{h}\left(1 - \frac{\epsilon}{1-\Delta}\right) < \theta$$

Again, given ϵ, Δ, and θ, the inequality holds if the value of h is sufficiently small. This shows that if θ and h are sufficiently small then the allocation where all organizations have three layers is indeed an equilibrium.

A.6 Proof of Proposition 8

I show that workers' wages fall as ϕ rises. Consider the following derivative of the equilibrium wage function

$$\frac{\partial w_1(x_1;\phi)}{\partial \phi} = \frac{\partial a(x_1;\phi)}{\partial \phi} + \frac{\partial C_k}{\partial \phi}(1-x_1) \tag{9}$$

where $x_1 \in [\theta, \theta+\phi]$ and C_k, $k = 1, 2, 3$ is a function of parameters defined in Appendix B.1. The first term in (9) is the change in the assignment function as an increase in ϕ raises the competition among workers and machines. The second term ensures that the wage function is continuous after an increase in ϕ. Note that

$$\frac{\partial a(x_1;\phi)}{\partial \phi} < 0$$
$$\frac{\partial C_k}{\partial \phi} > 0$$

I show that the first term dominates the second term if θ is sufficiently high. To see this, consider the case with $k = 1$. Then

$$\frac{\partial w_1(x_1;\phi)}{\partial \phi} = \frac{\partial z}{\partial \phi}\left[1 - \frac{(1-z+1/h) + hz(1-z+1/h) + \mu\Delta z + \hat{x}_2 - (1-\Delta) - \mu\Delta\theta}{(1-z+1/h)^2} \cdot (1-x_1)\right]$$

The goal is to show that the second term in the square brackets is smaller than one. Note that the denominator is increasing in θ and the numerator is decreasing in θ. Also, $1 - x_1 < 1 - \theta$ and so there are values of θ sufficiently large that deliver the desired inequality. This proves the proposition since a fall in workers' wages directly raises managers' wages.

A.7 Proof of Proposition 9

To show that more skilled managers gain more from technological change, note that

$$\frac{\partial w_2(x_2;\phi)}{\partial x_2} = \frac{1}{h(1-a^{-1}(x_2))}$$

by the envelope theorem. It suffices to show that the above derivative itself is increasing in ϕ since

$$\begin{aligned}\frac{\partial^2 w_2(x_2;\phi)}{\partial x_2 \partial \phi} =& \frac{\partial^2 w_2(x_2;\phi)}{\partial \phi \partial x_2} \\ =& \frac{1}{h(1-x_1)^2} \cdot \frac{\partial x_1}{\partial \phi}\end{aligned}$$

Note that since $\frac{\partial z}{\partial \phi} < 0$, it follows that $\frac{\partial a(x_1;\phi)}{\partial \phi} < 0$. By the implicit function theorem,

$$\frac{\partial x_1}{\partial \phi} > 0$$

which implies $\frac{\partial^2 w_2(x_2;\phi)}{\partial x_2 \partial \phi} > 0$ and proves the last statement of the proposition.

A.8 Proof of Corollary 10

Denote the total wages of top p earners by T_p. Then

$$T_p \equiv \int_{1-p}^{p} w_2(x_2) dx_2$$

For the income share of top p earners to rise with ϕ, the growth rate of T_p must be higher than the growth rate of total income in the economy. A sufficient condition for a higher growth rate of T_p is

$$\frac{\partial^2 w_2(x_2;\phi)}{\partial x_2 \partial \phi} > \frac{\partial w_2(x_2;\phi)}{\partial x_2}$$

That is, if the increase in $w_2(\cdot)$ due to an increase in ϕ grows faster with x_2 than $w_2(\cdot)$ itself, then the growth rate is increasing in x_2. Note that, from the proof of Proposition 9,

$$\frac{\partial^2 w_2(x_2;\phi)}{\partial x_2 \partial \phi} = \frac{\partial w_2(x_2;\phi)}{\partial x_2} \cdot \frac{1}{1-x_1} \cdot \frac{1-(\theta+\phi)}{1-x_1}$$

Since $1/(1-x_1) > 1$, I have the desired inequality for $x_1 = \theta + \phi$. Moreover, $\frac{1-(\theta+\phi)}{1-x_1}$ is continuous in x_1 so there is a left neighborhood of $\theta + \phi$ such that the inequality holds. This proves that the income share of top p agents is increasing in ϕ for small $p > 0$.

A.9 Proof of Proposition 11

Note that all existing machines' wages must fall because the supply of machines increases with an increase in the parameter ϕ. In an equilibrium where all machines are supervised by workers, higher ϕ allows workers to supervise more advanced machines and thus increase the mass of problems that are drawn and passed to their managers. As a result, there need to be more managers in the new equilibrium to solve more problems, which implies lower z.

I show that more skilled managers gain less from an increase in ϕ than less skilled managers. To do so, consider the cross derivative $\frac{\partial w_2(x_2;\phi)}{\partial\phi\partial x_2}$. By the envelop theorem, I have

$$\frac{\partial w_2(x_2;\phi)}{\partial x_2} = \frac{1}{h(1-x_1)}$$

Then, it follows that

$$\frac{\partial^2 w_2(x_2;\phi)}{\partial\phi\partial x_2} = \frac{1}{h(1-x_1)^2}\frac{\partial e(x_2;\phi)}{\partial\phi}$$

where $e(x_2;\phi)$ is the employment function, or an inverse of the assignment function. Since z decreases in the new equilibrium, there are fewer workers as the most skilled workers become managers. Then it must be that managers supervise less skilled workers, which implies $\frac{\partial e(x_2;\phi)}{\partial\phi} < 0$. This shows that $\frac{\partial^2 w_2(x_2;\phi)}{\partial\phi\partial x_2} < 0$ and the proposition.

A.10 Proof of Lemma 12

Assume $z < \theta$ is true and solve for an equilibrium. Then verify that $z < \theta$ is indeed true for sufficiently high values of θ. See Appendix B.4 for details.

A.11 Proof of Lemma 13

Solve for the assignment function assuming $z < \theta$ taking z, $\underline{y}$, and $\bar{y}$ as given. Then find the thresholds using the continuity conditions on the assignment function. See Appendix B.4 for details.

A.12 Proof of Proposition 15

For the first statement, I show

$$\frac{\partial z}{\partial h_m} < 0$$

The occupational threshold z is pinned down by the continuity conditions of the assignment function

$$\begin{aligned} a(1-\Delta) =& z \\ a(\underline{y}) =& \theta \\ a(\bar{y}) =& \theta + \phi \end{aligned}$$

As I show in Appendix B.4, z is the solution to the following equation that combines the conditions

$$F \equiv -\frac{h}{2}\left[z\frac{2}{h} + \Delta^2 - \frac{2}{h}\theta - \phi\frac{2}{\Phi}\right] + \frac{h}{2}(1-z)^2 + 1 - \theta - \phi = 0 \tag{10}$$

Then

$$\frac{\partial F}{\partial z} = -1 - h(1-z) < 0$$

$$\frac{\partial F}{\partial h_m} = -\frac{h}{2} \cdot \left(\phi\frac{2}{\Phi^2}\right)\frac{\partial \Phi}{\partial h_m} = -\frac{\phi h}{\Phi^2}\frac{\partial \Phi}{\partial h_m} < 0$$

Recall that $\Phi \equiv \frac{1/\Delta}{1/h\Delta + \mu/h_m}$ and so

$$\begin{aligned} \Phi =& \frac{\frac{1}{\Delta}}{\frac{1}{h\Delta} + \frac{\mu}{h_m}} = \frac{\frac{h_m}{\Delta}}{\frac{h_m}{h\Delta} + \mu} \\ \frac{\partial \Phi}{\partial h_m} =& \frac{\frac{1}{\Delta}}{\frac{h_m}{h\Delta} + \mu} - \frac{\frac{h_m}{\Delta}}{\left(\frac{h_m}{h\Delta} + \mu\right)^2}\frac{1}{h\Delta} = \frac{\frac{1}{\Delta}}{\frac{h_m}{h\Delta} + \mu}\left(1 - \frac{\frac{h_m}{h\Delta}}{\left(\frac{h_m}{h\Delta} + \mu\right)^2}\right) > 0 \end{aligned}$$

Thus, by the implicit function theorem I have

$$\frac{\partial z}{\partial h_m} = -\frac{\partial F/\partial h_m}{\partial F/\partial z} < 0$$

To see that z converges to θ as h_m falls, suppose $z = \theta$. Then the corresponding value of h_m is

$$h_m = \underline{h} \equiv \frac{h\phi\mu\Delta}{\frac{h}{2}\Delta^2 - \frac{h}{2}(1-\theta)^2 + \theta - 1} \tag{11}$$

which requires Δ and θ to be sufficiently large to be positive. Denote the above expression by $\underline{h}$. It follows that as $h_m \to \underline{h}$ $z \to \theta$. Moreover, from the continuity of the assignment function $\underline{y}$ is

$$\underline{y} = 1 - \sqrt{z\frac{2}{h} + \Delta^2 - \frac{2}{h}\theta}$$

which implies that $\underline{y} \to 1 - \Delta$ as $h_m \to \underline{y}$.

A.13 Proof of Lemma 14

Using (10), derive

$$\frac{\partial F}{\partial \phi} = \frac{\partial}{\partial \phi}\left(\frac{h\phi}{\Phi}\right) - 1 = \frac{h}{\Phi} - 1 = 1 + \frac{\mu\Delta}{h_m} - 1 = \frac{\mu\Delta}{h_m} > 0$$

Thus, by the implicit function theorem

$$\frac{\partial z}{\partial \phi} = -\frac{\partial F/\partial \phi}{\partial F/\partial z} > 0$$

which proves the proposition.

B Additional Derivations

B.1 Two Layers

In the following discussion, I lay out the steps to solve for an equilibrium.

1. Given the thresholds, solve for the assignment functions each of which satisfies the labor market clearing equation on each interval.
2. Given the thresholds and the assignment functions, solve for workers' and machines' wages using the first-order conditions.
3. Pin down the thresholds using the assignment functions and the wage functions.

Assignment Assuming $\theta < 1 - \Delta$, there are three intervals on which assignment functions and workers' wage functions are defined. First, $\mathcal{I}_1 = [\theta, 1 - \Delta]$ is the machine-only region. Second, $\mathcal{I}_2 = [1 - \Delta, z]$ is where workers and machines co-exist. Third, $\mathcal{I}_3 = [z, \theta + \phi]$ is populated only by machines that are more productive than any worker.

On $\mathcal{I}_1$, the labor market clearing equation is given by

$$g(x_1) = \frac{1}{h(1 - x_1)} f(a(x_1))a'(x_1), \;\; x_1 \in \mathcal{I}_1$$

On $\mathcal{I}_2$, however, managers face a larger pool of workers to choose from since there are now machines as well. In this case, the labor market clearing equation takes the following form:

$$(f(x_1) + g(x_1)) = \frac{1}{h(1 - x_1)} f(a(x_1))a'(x_1), \;\; x_1 \in \mathcal{I}_2$$

As is clear from the above equation, if there were no machines ($\mu = 0$) then the labor market clearing equation would be the same as the first expression. A positive value of μ implies that there is a greater supply of workers that pass on unsolved problems, and thus greater competition for managers. Lastly, the interval $\mathcal{I}_3$ is where advanced machines are. On this interval, the assignment function connects machines with top managers since these machines are the most skilled workers available to managers.

$$g(x_1) = \frac{1}{h(1 - x_1)} f(a(x_1))a'(x_1), \;\; x_1 \in \mathcal{I}_3$$

where, again, the right-hand side is the demand of managers for workers.

Since $f(x) = 1/\Delta$ and $g(x) = \mu$ over the relevant supports, the labor market clearing conditions become:

$$a'(x_1) = \mu\Delta h(1 - x_1), \;\; x_1 \in \mathcal{I}_1$$

$$a'(x_1) = (1 + \mu\Delta)h(1 - x_1), \;\; x_1 \in \mathcal{I}_2$$

$$a'(x_1) = \mu\Delta h(1 - x_1), \;\; x_1 \in \mathcal{I}_3$$

These are separate differential equations that can be solved independently given boundary conditions. To guarantee the continuity of the entire assignment function and market clearing over the entire interval $\mathcal{I}$, I impose the following boundary conditions:

$$a_1(\theta) = z$$
$$a_2(z) = \hat{x}_2$$
$$a_3(\theta+\phi) = 1$$

Given the thresholds, these conditions guarantee that the assignment function is one-to-one and onto as a mapping between $\mathcal{I}$ and $[z, 1]$. The equilibrium assignment function given the thresholds is then

$$a(x_1) = \begin{cases} a_1(x_1) = -\frac{\mu\Delta h}{2}(1-x_1)^2 + \frac{\mu\Delta h}{2}(1-\theta)^2 + z, \;\; x_1 \in \mathcal{I}_1 \\ a_2(x_1) = -\frac{(1+\mu\Delta)h}{2}(1-x_1)^2 + \frac{(1+\mu\Delta)h}{2}(1-z)^2 + \hat{x}_2, \;\; x_1 \in \mathcal{I}_2 \\ a_3(x_1) = -\frac{\mu\Delta h}{2}(1-x_1)^2 + \frac{\mu\Delta h}{2}(1-(\theta+\phi))^2 + 1, \;\; x_1 \in \mathcal{I}_3 \end{cases}$$

Additional conditions for continuity pin down the equilibrium thresholds

$$a_1(1-\Delta) = \tilde{x}_2 \implies \tilde{x}_2(z)$$
$$a_2(\theta) = \tilde{x}_2 \implies \hat{x}_2(z)$$
$$a_3(z) = \hat{x}_2 \implies z$$

which give

$$z = \frac{h+1-\sqrt{1+h^2\Delta^2+(1+\mu\Delta)h^2((1-\theta)^2-(1-(\theta+\phi))^2)}}{h}$$
$$\tilde{x}_2 = -\frac{\mu\Delta h}{2}\Delta^2 + \frac{\mu\Delta h}{2}(1-\theta)^2 + z$$
$$\hat{x}_2 = \tilde{x}_2 + \frac{(1+\mu\Delta)h}{2}(1-\theta)^2 - \frac{(1+\mu\Delta)h}{2}(1-z)^2$$

Consider the following comparative statics result

$$\frac{\partial a(x_1;\phi)}{\partial \phi} = \begin{cases} \frac{\partial z}{\partial \phi} < 0, \;\; x_1 \in \mathcal{I}_1 \\ -(1+\mu\Delta)h(1-z)\frac{\partial z}{\partial \phi} + \frac{\partial \hat{x}_2}{\partial \phi} = \frac{\partial z}{\partial \phi} < 0, \;\; x_1 \in \mathcal{I}_2 \\ -\mu\Delta h(1-(\theta+\phi)) < 0, \;\; x_1 \in \mathcal{I}_3 \end{cases}$$

Wages The equilibrium wage function is determined by the first-order condition

$$w_1'(x_1) = \frac{x_2 - w_1(x_1)}{1 - x_1}$$

The general solution to the above differential equation is

$$w_1(x_1) = \begin{cases} a(x_1) - \mu\Delta h x_1(1-x_1) + C_1(1-x_1), \;\; x_1 \in \mathcal{I}_1 \\ a(x_1) - (1+\mu\Delta)h x_1(1-x_1) + C_2(1-x_1), \;\; x_1 \in \mathcal{I}_2 \\ a(x_1) - \mu\Delta h x_1(1-x_1) + C_3(1-x_1), \;\; x_1 \in \mathcal{I}_3 \end{cases}$$

where C_k, $k = 1, 2, 3$, is some constant. Workers' wages must be continuous at $1 - \Delta$ and z. Moreover, the marginal managers must earn the same amount as the marginal workers. Thus, the constants of integration must satisfy

$$\begin{aligned} a(1-\Delta) - \mu\Delta h(1-\Delta)\Delta + C_1\Delta =& a(1-\Delta) - (1+\mu\Delta)h(1-\Delta)\Delta + C_2\Delta \\ a(z) - (1+\mu\Delta)hz(1-z) + C_2(1-z) =& a(x_1) - \mu\Delta hz(1-z) + C_3(1-z) \\ a(z) - \mu\Delta hz(1-z) + C_3(1-z) =& w_2(z) \end{aligned}$$

The three conditions pin down the constants:

$$\begin{aligned} C_1 =& \frac{\mu\Delta\theta - \hat{x}_2 + (1+\mu\Delta)hz(1-z) - h(1-\Delta)(1-z)}{1 - z + 1/h} \\ C_2 =& C_1 + h(1-\Delta) \\ C_3 =& C_2 - hz \end{aligned}$$

To derive $\frac{\partial w_1(a^{-1}(x_2);\phi)}{\partial \phi}$, take the derivative of C_1 with respect to ϕ

$$
\begin{aligned}
\frac{dC_1}{d\phi} =& \frac{1}{1-z+1/h}\left(-\frac{\partial \hat{x}_2}{\partial \phi} + (1+\mu\Delta)h(1-2z)\frac{\partial z}{\partial \phi} + h(1-\Delta)\frac{\partial z}{\partial \phi}\right) \\
&+ \frac{\mu\Delta\theta - \hat{x}_2 + \mu\Delta h z(1-z) - h(1-\Delta)(1-z)}{(1-z+1/h)^2}\frac{\partial z}{\partial \phi}
\end{aligned}
$$

After rearranging the terms, it follows that

$$
\begin{aligned}
\frac{dC_1}{d\phi} =& \frac{\partial z}{\partial \phi}\frac{1}{(1-z+1/h)^2} \\
&\times [-(1-z+1/h) - \mu\Delta z - hz(1-z+1/h) + (1-\Delta) + \mu\Delta\theta - \hat{x}_2] > 0
\end{aligned}
$$

B.2 Three Layers

Assignment To begin with, the equilibrium assignment functions solve the following differential equations

$$
\begin{aligned}
g(x_0) &= \frac{1}{h(1-x_0)} f(\tilde{a}(x_0))\tilde{a}'(x_0), \;\; x_0 \in [\theta^*, \theta+\phi] \\
f(x_1) &= \frac{1-x_0}{1-x_1} f(a(x_1))a'(x_1), \;\; x_1 \in [1-\Delta, z]
\end{aligned}
$$

where $\tilde{a}$ is the assignment function that connects machines and workers, and a is the assignment function that connects workers and managers. Assuming $g(x) = \mu$ and $f(x) = 1/\Delta$ as before, and imposing the terminal condition $\tilde{a}(\theta+\phi) = z$, I have

$$
\tilde{a}(x_0) = -\frac{\mu\Delta h}{2}(1-x_0)^2 + \frac{\mu\Delta h}{2}(1-\theta-\phi)^2 + z, \;\; x_0 \in [\theta^*, \theta+\phi]
$$

Note that the employment function is

$$
1 - \tilde{e}(x_1) = 1 - \tilde{a}^{-1}(x_1) = \sqrt{(1-\theta-\phi)^2 + (z-x_1)/\Phi}, \;\; x_1 \in [1-\Delta, z]
$$

where $\Phi \equiv \mu\Delta h/2$. The equilibrium assignment function $a(\cdot)$ from workers to managers is:

$$\begin{aligned} a(x_1) =& 2\Phi\sqrt{(1-\theta-\phi)^2 + (t-z)/\Phi} \\ & \times \left[\left(\frac{2}{3}\frac{1}{\Phi} - 1 \right) t + 1 + \frac{2}{3}(1-\theta-\phi)^2 - \frac{2}{3}\frac{1}{\Phi} z \right] \Bigg|_{1-\Delta}^{x_1} + z \end{aligned}$$

where t is a dummy variable.

The market for machines clears through adjustments in the entry threshold θ^* that is determined by the free entry condition.

Lemma 17 (Entry threshold)**.** *There exists a cutoff skill level $\theta^* \in [\theta, \theta + \phi]$ such that $w_0(\theta^*) = \epsilon$. Thus, marginal machine producer earns zero profits in equilibrium.*

Proof. If $w_0(\theta^*) > \epsilon$ in equilibrium then machine producers with lower skills can enter and θ^* must fall. If $w_0(\theta^*) < \epsilon$ then the opposite holds. □

Furthermore, the occupational threshold z is pinned down by the condition $a(z) = 1$.

Wages Given the assignment functions and the thresholds, I can find the wage functions by solving the system of differential equations. The boundary conditions are

$$\begin{aligned} w_0(\theta^*) &= \epsilon \\ w_1(z) &= w_2(z) \end{aligned}$$

The first condition states that the least skilled machines earn zero net income. This is because managers can lower the machine rent down to the entry cost and still hire them. The second condition ensures that marginal managers are indifferent to becoming workers.

$$\begin{aligned} w_2(x_2) &= \max_{x_1 \in [1-\Delta, z]} n_1 n_0 \left(x_2 - x_1 + p(x_1) \right) \\ w_1(x_1) &= \max_{x_0 \in [\theta, \theta+\phi]} n_0 \left(x_1 - x_0 + p(x_0) - p(x_1) \right) \end{aligned}$$

where

$$p(x_1) = x_1 - x_0 - \frac{1}{n_0} w_1(x_1) + p(x_0)$$
$$p(x_0) = x_0 - w_0(x_0)$$

Substitute $n_0 = \frac{1}{h(1-x_0)}$ and $n_1 = \frac{1}{h(1-x_1)}$ into the problem.

$$w_2(x_2) = \max_{x_1 \in [1-\Delta, z]} \frac{x_2 - x_1 + p(x_1)}{h(1-x_1)}$$
$$w_1(x_1) = \max_{x_0 \in [\theta, \theta+\phi]} \frac{x_1 - x_0 + p(x_0) - p(x_1)}{h(1-x_0)}$$

where

$$p(x_1) = x_1 - x_0 - h(1-x_0) w_1(x_1) + p(x_0)$$
$$p(x_0) = x_0 - w_0(x_0)$$

B.3 Solving the Model under Alternative Parameterizations

To verify the uniqueness of an equilibrium at each point on the parameter space, I calculate the allocation and see whether agents have an incentive to deviate from the allocation. To do so, I solve the allocation by imposing different boundary conditions and checking which boundary conditions are appropriate.

There are several cases to consider for both the two-layer and three-layer cases. The procedure follows the steps below:

1. Calculate thresholds for all possible cases.
2. Determine which case is valid.
3. Solve for the assignment function and the wage function.
4. Check the incentive to deviate.

B.3.1 Two Layers

Case 1: $\theta < \theta + \phi < 1 - \Delta < z$

$$\mathcal{I}_1 = [\theta, \theta + \phi]$$
$$\mathcal{I}_2 = [1 - \Delta, z]$$

$$g(x_1) = \frac{1}{h(1 - x_1)} f(a(x_1)) a'(x_1), \;\; x_1 \in \mathcal{I}_1$$
$$f(x_1) = \frac{1}{h(1 - x_1)} f(a(x_1)) a'(x_1), \;\; x_1 \in \mathcal{I}_2$$

Since $f(x) = 1/\Delta$ and $g(x) = \mu$ over the relevant supports, the labor market clearing conditions become:

$$a'(x_1) = \mu \Delta h(1 - x_1), \;\; x_1 \in \mathcal{I}_1$$
$$a'(x_1) = h(1 - x_1), \;\; x_1 \in \mathcal{I}_2$$

To guarantee the continuity of the entire assignment function and market clearing over the entire interval $\mathcal{I}$, I impose the following boundary conditions:

$$a_1(\theta) = z$$
$$a_2(z) = 1$$

Given the thresholds, these conditions guarantee that the assignment function is one-to-one and onto as a mapping between $\mathcal{I}$ and $[z, 1]$. The equilibrium assignment function given the thresholds is then

$$a(x_1) = \begin{cases} a_1(x_1) = -\frac{\mu \Delta h}{2}(1 - x_1)^2 + \frac{\mu \Delta h}{2}(1 - \theta)^2 + z, \;\; x_1 \in \mathcal{I}_1 \\ a_2(x_1) = -\frac{h}{2}(1 - x_1)^2 + \frac{h}{2}(1 - z)^2 + 1, \;\; x_1 \in \mathcal{I}_2 \end{cases}$$

Additional conditions for continuity pin down the equilibrium thresholds

$$a_1(\theta+\phi) = \hat{x}_2 \implies \hat{x}_2(z)$$
$$a_2(1-\Delta) = \hat{x}_2 \implies z$$

which give

$$z = 1 + 1/h - \sqrt{\Delta^2 + 1/h^2 + \mu\Delta\left[(1-\theta)^2 - (1-\theta-\phi)^2\right]}$$
$$\hat{x}_2 = -\frac{\mu\Delta h}{2}(1-\theta-\phi)^2 + \frac{\mu\Delta h}{2}(1-\theta)^2 + z$$

$$w_1(x_1) = \begin{cases} w_{11}(x_1) = a_1(x_1) - \mu\Delta h x_1(1-x_1) + C_1(1-x_1), & x_1 \in \mathcal{I}_1 \\ w_{12}(x_1) = a_2(x_1) - \mu\Delta h x_1(1-x_1) + C_2(1-x_1), & x_1 \in \mathcal{I}_2 \end{cases}$$

Using the conditions $w_{21}(\hat{x}_2) = w_{22}(\hat{x}_2)$ and $w_{12}(z) = w_2(z)$, the constants are

$$C_2 = \frac{\mu\Delta(1-\Delta-\phi) - 1 + \mu\Delta h z(1-z)}{1 - z + 1/h}$$
$$C_1 = C_2 + \mu\Delta h((\theta+\phi) - (1-\Delta))$$

Case 2: $\theta < 1-\Delta < \theta+\phi < z$

$$\mathcal{I}_1 = [\theta, 1-\Delta]$$
$$\mathcal{I}_2 = [1-\Delta, \theta+\phi]$$
$$\mathcal{I}_3 = [\theta+\phi, z]$$

$$g(x_1) = \frac{1}{h(1-x_1)} f(a(x_1))a'(x_1), \ \ x_1 \in \mathcal{I}_1$$
$$f(x_1) + g(x_1) = \frac{1}{h(1-x_1)} f(a(x_1))a'(x_1), \ \ x_1 \in \mathcal{I}_2$$
$$f(x_1) = \frac{1}{h(1-x_1)} f(a(x_1))a'(x_1), \ \ x_1 \in \mathcal{I}_2$$

Since $f(x) = 1/\Delta$ and $g(x) = \mu$ over the relevant supports, the labor market clearing conditions become:

$$a'(x_1) = \mu\Delta h(1 - x_1), \;\; x_1 \in \mathcal{I}_1$$
$$a'(x_1) = (1 + \mu\Delta)h(1 - x_1), \;\; x_1 \in \mathcal{I}_2$$
$$a'(x_1) = h(1 - x_1), \;\; x_1 \in \mathcal{I}_3$$

To guarantee the continuity of the entire assignment function and market clearing over the entire interval $\mathcal{I}$, I impose the following boundary conditions:

$$a_1(\theta) = z$$
$$a_2(\theta + \phi) = \hat{x}_2$$
$$a_3(z) = 1$$

Given the thresholds, these conditions guarantee that the assignment function is one-to-one and onto as a mapping between $\mathcal{I}$ and $[z, 1]$. The equilibrium assignment function given the thresholds is then

$$a(x_1) = \begin{cases} a_1(x_1) \equiv -\frac{\mu\Delta h}{2}(1 - x_1)^2 + \frac{\mu\Delta h}{2}(1 - \theta)^2 + z, \;\; x_1 \in \mathcal{I}_1 \\ a_2(x_1) \equiv -\frac{(1+\mu\Delta)h}{2}(1 - x_1)^2 + \frac{(1+\mu\Delta)h}{2}(1 - \theta - \phi)^2 + \hat{x}_2, \;\; x_1 \in \mathcal{I}_2 \\ a_3(x_1) \equiv -\frac{h}{2}(1 - x_1)^2 + \frac{h}{2}(1 - z)^2 + 1, \;\; x_1 \in \mathcal{I}_3 \end{cases}$$

Additional conditions for continuity pin down the equilibrium thresholds

$$a_1(1 - \Delta) = a_2(1 - \Delta) \implies \hat{x}_2(z)$$
$$a_2(\theta + \phi) = a_3(\theta + \phi) \implies z$$

which give

$$
\begin{aligned}
z =& 1 + 1/h - \sqrt{1/h^2 + \Delta^2 + \mu\Delta[(1-\theta)^2 - (1-\theta-\phi)^2]} \\
\hat{x}_2 =& -\frac{h}{2}(1-\theta-\phi)^2 + \frac{h}{2}(1-z)^2 + 1
\end{aligned}
$$

$$
w_1(x_1) = \begin{cases}
w_{11}(x_1) = a_1(x_1) - \mu\Delta h x_1(1-x_1) + C_1(1-x_1), \;\; x_1 \in \mathcal{I}_1 \\
w_{12}(x_1) = a_2(x_1) - (1+\mu\Delta)h x_1(1-x_1) + C_2(1-x_1), \;\; x_1 \in \mathcal{I}_2 \\
w_{13}(x_1) = a_3(x_1) - h x_1(1-x_1) + C_3(1-x_1), \;\; x_1 \in \mathcal{I}_3
\end{cases}
$$

Using the continuity conditions $w_{11}(1-\Delta) = w_{12}(1-\Delta)$, $w_{12}(\theta+\phi) = w_{13}(\theta+\phi)$, and $w_{13}(z) = w_2(z)$, the constants are

$$
\begin{aligned}
C_1 =& C_2 - h(1-\Delta) \\
C_2 =& C_3 + \mu\Delta h(\theta+\phi) \\
C_3 =& \frac{\mu\Delta h - \mu\Delta(\theta+\phi) - \Delta + hz(1-z)}{1 - z + 1/h}
\end{aligned}
$$

Case 3: $\theta < 1-\Delta < z < \theta+\phi$ This is the original case.

$$
\begin{aligned}
z =& \frac{h + 1 - \sqrt{1 + h^2\Delta^2 + (1+\mu\Delta)h^2((1-\theta)^2 - (1-(\theta+\phi))^2)}}{h} \\
=& 1 + 1/h - \sqrt{1/h^2 + \Delta^2 + (1+\mu\Delta)[(1-\theta)^2 - (1-(\theta+\phi))^2]} \\
\tilde{x}_2 =& -\frac{\mu\Delta h}{2}\Delta^2 + \frac{\mu\Delta h}{2}(1-\theta)^2 + z \\
\hat{x}_2 =& \tilde{x}_2 + \frac{(1+\mu\Delta)h}{2}(1-\theta)^2 - \frac{(1+\mu\Delta)h}{2}(1-z)^2
\end{aligned}
$$

$$
w_1(x_1) = \begin{cases}
a(x_1) - \mu\Delta h x_1(1-x_1) + C_1(1-x_1), \;\; x_1 \in \mathcal{I}_1 \\
a(x_1) - (1+\mu\Delta)h x_1(1-x_1) + C_2(1-x_1), \;\; x_1 \in \mathcal{I}_2 \\
a(x_1) - \mu\Delta h x_1(1-x_1) + C_3(1-x_1), \;\; x_1 \in \mathcal{I}_3
\end{cases}
$$

$$
\begin{aligned}
C_1 =& \frac{\mu\Delta\theta - \hat{x}_2 + (1+\mu\Delta)hz(1-z) - h(1-\Delta)(1-z)}{1-z+1/h} \\
C_2 =& C_1 + h(1-\Delta) \\
C_3 =& C_2 - hz
\end{aligned}
$$

Case 4: $1-\Delta < \theta < \theta+\phi < z$

$$
\begin{aligned}
\mathcal{I}_1 &= [1-\Delta, \theta] \\
\mathcal{I}_2 &= [\theta, \theta+\phi] \\
\mathcal{I}_3 &= [\theta+\phi, z]
\end{aligned}
$$

$$
\begin{aligned}
f(x_1) &= \frac{1}{h(1-x_1)} f(a(x_1))a'(x_1), \;\; x_1 \in \mathcal{I}_1 \\
f(x_1) + g(x_1) &= \frac{1}{h(1-x_1)} f(a(x_1))a'(x_1), \;\; x_1 \in \mathcal{I}_2 \\
f(x_1) &= \frac{1}{h(1-x_1)} f(a(x_1))a'(x_1), \;\; x_1 \in \mathcal{I}_2
\end{aligned}
$$

Since $f(x) = 1/\Delta$ and $g(x) = \mu$ over the relevant supports, the labor market clearing conditions become:

$$
\begin{aligned}
a'(x_1) &= h(1-x_1), \;\; x_1 \in \mathcal{I}_1 \\
a'(x_1) &= (1+\mu\Delta)h(1-x_1), \;\; x_1 \in \mathcal{I}_2 \\
a'(x_1) &= h(1-x_1), \;\; x_1 \in \mathcal{I}_3
\end{aligned}
$$

To guarantee the continuity of the entire assignment function and market clearing over the entire interval $\mathcal{I}$, I impose the following boundary conditions:

$$
\begin{aligned}
a(1-\Delta) &= z \\
a(\theta+\phi) &= \hat{x}_2 \\
a(z) &= 1
\end{aligned}
$$

Given the thresholds, these conditions guarantee that the assignment function is one-to-one and onto as a mapping between $\mathcal{I}$ and $[z, 1]$. The equilibrium assignment function given the thresholds is then

$$a(x_1) = \begin{cases} a_1(x_1) \equiv -\frac{h}{2}(1-x_1)^2 + \frac{h}{2}(1-\theta)^2 + z, \;\; x_1 \in \mathcal{I}_1 \\ a_2(x_1) \equiv -\frac{(1+\mu\Delta)h}{2}(1-x_1)^2 + \frac{(1+\mu\Delta)h}{2}(1-\theta-\phi)^2 + \hat{x}_2, \;\; x_1 \in \mathcal{I}_2 \\ a_3(x_1) \equiv -\frac{h}{2}(1-x_1)^2 + \frac{h}{2}(1-z)^2 + 1, \;\; x_1 \in \mathcal{I}_3 \end{cases}$$

Additional conditions for continuity pin down the equilibrium thresholds

$$a_1(\theta) = a_2(\theta) \implies \hat{x}_2(z)$$
$$a_2(\theta+\phi) = a_3(\theta+\phi) \implies z$$

which give

$$z = \frac{1 + h - \sqrt{1 + 2\Delta h^2\mu\phi - \Delta h^2\mu\phi^2 - 2\Delta h^2\mu\phi\theta}}{h}$$
$$\hat{x}_2 = z + \frac{(1+\mu\Delta)h}{2}(1-\theta)^2 - \frac{(1+\mu\Delta)h}{2}(1-\theta-\phi)^2$$

$$w_1(x_1) = \begin{cases} w_{11}(x_1) = a_1(x_1) - hx_1(1-x_1) + C_1(1-x_1), \;\; x_1 \in \mathcal{I}_1 \\ w_{12}(x_1) = a_2(x_1) - (1+\mu\Delta)hx_1(1-x_1) + C_2(1-x_1), \;\; x_1 \in \mathcal{I}_2 \\ w_{13}(x_1) = a_3(x_1) - hx_1(1-x_1) + C_3(1-x_1), \;\; x_1 \in \mathcal{I}_3 \end{cases}$$

Using the continuity conditions $w_{11}(\theta) = w_{12}(\theta)$, $w_{12}(\theta+\phi) = w_{13}(\theta+\phi)$, and $w_{13}(z) = w_2(z)$, the constants are

$$C_1 = C_2 - \mu\Delta h\theta$$
$$C_2 = C_3 + \mu\Delta h(\theta+\phi)$$
$$C_3 = \frac{hz(1-z) - \Delta - \mu\Delta\phi}{1 - z + 1/h}$$

Case 5: $1 - \Delta < \theta < z < \theta + \phi$

$$\mathcal{I}_1 = [1 - \Delta, \theta]$$
$$\mathcal{I}_2 = [\theta, z]$$
$$\mathcal{I}_3 = [z, \theta + \phi]$$

$$f(x_1) = \frac{1}{h(1-x_1)} f(a(x_1))a'(x_1), \;\; x_1 \in \mathcal{I}_1$$
$$f(x_1) + g(x_1) = \frac{1}{h(1-x_1)} f(a(x_1))a'(x_1), \;\; x_1 \in \mathcal{I}_2$$
$$g(x_1) = \frac{1}{h(1-x_1)} f(a(x_1))a'(x_1), \;\; x_1 \in \mathcal{I}_2$$

Since $f(x) = 1/\Delta$ and $g(x) = \mu$ over the relevant supports, the labor market clearing conditions become:

$$a'(x_1) = h(1 - x_1), \;\; x_1 \in \mathcal{I}_1$$
$$a'(x_1) = (1 + \mu\Delta)h(1 - x_1), \;\; x_1 \in \mathcal{I}_2$$
$$a'(x_1) = \mu\Delta h(1 - x_1), \;\; x_1 \in \mathcal{I}_3$$

To guarantee the continuity of the entire assignment function and market clearing over the entire interval $\mathcal{I}$, I impose the following boundary conditions:

$$a_1(1 - \Delta) = z$$
$$a_2(z) = \hat{x}_2$$
$$a_3(\theta + \phi) = 1$$

Given the thresholds, these conditions guarantee that the assignment function is one-to-one and onto as a mapping between $\mathcal{I}$ and $[z, 1]$. The equilibrium assignment function given the thresholds is then

$$a(x_1) = \begin{cases} a_1(x_1) \equiv -\frac{h}{2}(1-x_1)^2 + \frac{h}{2}\Delta^2 + z, \;\; x_1 \in \mathcal{I}_1 \\ a_2(x_1) \equiv -\frac{(1+\mu\Delta)h}{2}(1-x_1)^2 + \frac{(1+\mu\Delta)h}{2}(1-z)^2 + \hat{x}_2, \;\; x_1 \in \mathcal{I}_2 \\ a_3(x_1) \equiv -\frac{\mu\Delta h}{2}(1-x_1)^2 + \frac{\mu\Delta h}{2}(1-\theta-\phi)^2 + 1, \;\; x_1 \in \mathcal{I}_3 \end{cases}$$

Additional conditions for continuity pin down the equilibrium thresholds

$$a_1(\theta) = a_2(\theta) \implies \hat{x}_2(z)$$
$$a_2(z) = a_3(z) \implies z$$

which give

$$z = 1 + 1/h - \sqrt{\Delta^2 + 1/h^2 + \mu\Delta[(1-\theta)^2 - (1-\theta-\phi)^2]}$$
$$\hat{x}_2 = z + \frac{\mu\Delta h}{2}(1-\theta)^2 + \frac{h}{2}\Delta^2 - \frac{(1+\mu\Delta)h}{2}(1-z)^2$$

$$w_1(x_1) = \begin{cases} w_{11}(x_1) = a_1(x_1) - hx_1(1-x_1) + C_1(1-x_1), \;\; x_1 \in \mathcal{I}_1 \\ w_{12}(x_1) = a_2(x_1) - (1+\mu\Delta)hx_1(1-x_1) + C_2(1-x_1), \;\; x_1 \in \mathcal{I}_2 \\ w_{13}(x_1) = a_3(x_1) - \mu\Delta hx_1(1-x_1) + C_3(1-x_1), \;\; x_1 \in \mathcal{I}_3 \end{cases}$$

Using the continuity conditions $w_{11}(\theta) = w_{12}(\theta)$, $w_{12}(z) = w_{13}(z)$, and $w_{13}(z) = w_2(z)$, the constants are

$$C_1 = C_2 - \mu\Delta h\theta$$
$$C_2 = C_3 + hz$$
$$C_3 = \frac{1 - \Delta - z + \mu\Delta\theta - \hat{x}_2 + \mu\Delta hz(1-z)}{1 - z + 1/h}$$

B.3.2 Three Layers

Case 1: $\theta + \phi < 1 - \Delta$ To begin with, the equilibrium assignment functions solve the following differential equations

$$g(x_0) = \frac{1}{h(1-x_0)} f(\tilde{a}(x_0))\tilde{a}'(x_0), \;\; x_0 \in [\theta^*, \theta + \phi]$$
$$f(x_1) = \frac{1-x_0}{1-x_1} f(a(x_1))a'(x_1), \;\; x_1 \in [1-\Delta, z]$$

where $\tilde{a}$ is the assignment function that connects machines and workers, and a is the assignment function that connects workers and managers. Assuming $g(x) = \mu$ and $f(x) = 1/\Delta$ as before, and imposing the terminal condition $\tilde{a}(\theta + \phi) = z$, I have

$$\tilde{a}(x_0) = -\frac{\mu \Delta h}{2}(1-x_0)^2 + \frac{\mu \Delta h}{2}(1-\theta-\phi)^2 + z, \;\; x_0 \in [\theta^*, \theta + \phi]$$

Note that the employment function is

$$1 - \tilde{e}(x_1) = 1 - \tilde{a}^{-1}(x_1) = \sqrt{(1-\theta-\phi)^2 + (z-x_1)/\Phi}, \;\; x_1 \in [1-\Delta, z]$$

where $\Phi \equiv \mu \Delta h/2$. With the initial condition $a(1-\Delta) = z$, the assignment function from workers to managers is

$$\begin{aligned} a(x_1) =& 2\Phi\sqrt{(1-\theta-\phi)^2 + (t-z)/\Phi} \\ &\times \left[\left(\frac{2}{3}\frac{1}{\Phi} - 1 \right) t + 1 + \frac{2}{3}(1-\theta-\phi)^2 - \frac{2}{3}\frac{1}{\Phi} z \right] \Bigg|_{1-\Delta}^{x_1} + z \end{aligned}$$

where t is a dummy variable.

Case 2: $1 - \Delta < \theta + \phi < z_1 < z$

$$\begin{aligned} g(x_0) &= \frac{1}{h(1-x_0)} f(\tilde{a}(x_0))\tilde{a}'(x_0), \;\; x_0 \in [\theta^*, 1-\Delta] \\ f(x_1) &= \frac{1-x_0}{1-x_1} f(a(x_1))a'(x_1), \;\; x_1 \in [z_1, \tilde{x}_1] \\ \tilde{a}(1-\Delta) &= \tilde{x}_1 \end{aligned}$$

$$
\begin{aligned}
g(x_0) + f(x_0) &= \frac{1}{h(1-x_0)} f(\tilde{a}(x_0))\tilde{a}'(x_0),\;\; x_0 \in [1-\Delta, \theta+\phi] \\
f(x_1) &= \frac{1-x_0}{1-x_1} f(a(x_1))a'(x_1),\;\; x_1 \in [\tilde{x}_1, \hat{x}_1] \\
\tilde{a}(\theta+\phi) &= \hat{x}_1
\end{aligned}
$$

$$
\begin{aligned}
f(x_0) &= \frac{1}{h(1-x_0)} f(\tilde{a}(x_0))\tilde{a}'(x_0),\;\; x_0 \in [\theta+\phi, z_1] \\
f(x_1) &= \frac{1-x_0}{1-x_1} f(a(x_1))a'(x_1),\;\; x_1 \in [\hat{x}_1, z] \\
\tilde{a}(z_1) &= z
\end{aligned}
$$

$$
\tilde{a}(x_0) = \begin{cases}
\tilde{a}_1(x_0) \equiv -\frac{\mu\Delta h}{2}(1-x_0)^2 + \frac{\mu\Delta h}{2}\Delta^2 + \tilde{x}_1,\;\; x_0 \in [\theta^*, 1-\Delta] \\
\tilde{a}_2(x_0) \equiv -\frac{(1+\mu\Delta)h}{2}(1-x_0)^2 + \frac{(1+\mu\Delta)h}{2}(1-\theta-\phi)^2 + \hat{x}_1,\;\; x_0 \in [1-\Delta, \theta+\phi] \\
\tilde{a}_3(x_0) \equiv -\frac{h}{2}(1-x_0)^2 + \frac{h}{2}(1-z_1)^2 + z,\;\; x_0 \in [\theta+\phi, z_1]
\end{cases}
$$

$$
a'(x_1) = \begin{cases}
\frac{1-x_1}{\sqrt{\Delta^2 + \frac{2}{\mu\Delta h}(\tilde{x}_1 - x_1)}},\;\; x_1 \in [z_1, \tilde{x}_1],\;\; a(z_1) = z \\
\frac{1-x_1}{\sqrt{(1-\theta-\phi)^2 + \frac{2}{(1+\mu\Delta)h}(\hat{x}_1 - x_1)}},\;\; x_1 \in [\tilde{x}_1, \hat{x}_1],\;\; a(\tilde{x}_1) = \tilde{x}_2 \\
\frac{1-x_1}{\sqrt{(1-z_1)^2 + \frac{2}{h}(z - x_1)}},\;\; x_1 \in [\hat{x}_1, z],\;\; a(\hat{x}_1) = \hat{x}_2
\end{cases}
$$

Assignment function 1: The solution to the given differential equation

$$
a'(x_1) = \frac{1-x_1}{\sqrt{\Delta^2 + \frac{2}{\mu\Delta h}(\tilde{x}_1 - x_1)}}
$$

with the boundary condition $a(z_1) = z$ and $x_1 \in [z_1, \tilde{x}_1]$ is:

$$
a(x_1) = \frac{1}{3}\left(-3AB + A^2B + 2A\tilde{x}_1B + Ax_1B + 3AC - A^2C - 2A\tilde{x}_1C - Az_1C + 3z\right)
$$

where

$$\begin{aligned} A &= \Delta h\mu, \\ B &= \sqrt{\frac{\Delta^3 h\mu + 2\tilde{x}_1 - 2x_1}{\Delta h\mu}}, \\ C &= \sqrt{\frac{\Delta^3 h\mu + 2\tilde{x}_1 - 2z_1}{\Delta h\mu}}. \end{aligned}$$

Assignment function 2: The general solution to the given differential equation

$$a'(x_1) = \frac{1 - x_1}{\sqrt{(1-\theta-\phi)^2 + \frac{2}{(1+\mu\Delta)h}(\hat{x}_1 - x_1)}}$$

is:

$$\begin{aligned} a(x_1) &= \frac{DEF}{3} + C_1 \\ \text{where} \quad D &= h(1+\Delta\mu), \\ E &= \sqrt{\frac{D(-1+\phi+\theta)^2 + 2(\hat{x}_1 - x_1)}{h + \Delta h\mu}}, \\ F &= -3 + 2\hat{x}_1 + D(-1+\phi+\theta)^2 + x_1. \end{aligned}$$

Here, C_1 is the constant of integration, which can be determined using the boundary condition $a(\tilde{x}_1) = \tilde{x}_2$. The constant C_1 can be determined using the boundary condition $a(\tilde{x}_1) = \tilde{x}_2$ as follows:

$$\begin{aligned} C_1 = -\frac{1}{3}\Bigg(& h(1+\Delta\mu)\sqrt{\frac{h(1+\Delta\mu)(-1+\phi+\theta)^2 + 2(\hat{x}_1 - \tilde{x}_1)}{h + \Delta h\mu}} \\ & \times (-3 + 2\hat{x}_1 + h(1+\Delta\mu)(-1+\phi+\theta)^2 + \tilde{x}_1)\Bigg) \\ & + \tilde{x}_2 \end{aligned}$$

With this constant, you can fully specify the function $a(x_1)$ given the boundary condition. **As-**

signment function 3: The solution to the given differential equation

$$a'(x_1) = \frac{1 - x_1}{\sqrt{(1 - z_1)^2 + \frac{2}{h}(z - x_1)}}$$

with the boundary condition $a(\hat{x}_1) = \hat{x}_2$ and $x_1 \in [\hat{x}_1, z]$ is:

$$\begin{aligned} a(x_1) = \frac{1}{3}\bigg(& 3\hat{x}_2 + 3hG - h^2G - h\hat{x}_1G + 2h^2z_1G - h^2z_1^2G \\ & - 2hzG - 3hH + h^2H + hx_1H - 2h^2z_1H + h^2z_1^2H + 2hzH \bigg) \end{aligned}$$

$$\text{where} \quad G = \sqrt{\frac{h - 2\hat{x}_1 - 2hz_1 + hz_1^2 + 2z}{h}}, \qquad H = \sqrt{\frac{h - 2x_1 - 2hz_1 + hz_1^2 + 2z}{h}}.$$

Case 3: $1 - \Delta < z_1 < \theta + \phi < z$

$$\begin{aligned} g(x_0) &= \frac{1}{h(1 - x_0)} f(\tilde{a}(x_0))\tilde{a}'(x_0), \;\; x_0 \in [\theta, 1 - \Delta] \\ f(x_1) &= \frac{1 - x_0}{1 - x_1} f(a(x_1))a'(x_1), \;\; x_1 \in [z_1, \tilde{x}_1] \\ \tilde{a}(1 - \Delta) &= \tilde{x}_1 \end{aligned}$$

$$\begin{aligned} g(x_0) + f(x_0) &= \frac{1}{h(1 - x_0)} f(\tilde{a}(x_0))\tilde{a}'(x_0), \;\; x_0 \in [1 - \Delta, z_1] \\ f(x_1) &= \frac{1 - x_0}{1 - x_1} f(a(x_1))a'(x_1), \;\; x_1 \in [\tilde{x}_1, \hat{x}_1] \\ \tilde{a}(z_1) &= \hat{x}_1 \end{aligned}$$

$$\begin{aligned} f(x_0) &= \frac{1}{h(1 - x_0)} f(\tilde{a}(x_0))\tilde{a}'(x_0), \;\; x_0 \in [z_1, \theta + \phi] \\ f(x_1) &= \frac{1 - x_0}{1 - x_1} f(a(x_1))a'(x_1), \;\; x_1 \in [\hat{x}_1, z] \\ \tilde{a}(\theta + \phi) &= z \end{aligned}$$

$$\tilde{a}(x_0) = \begin{cases} \tilde{a}_1(x_0) \equiv -\frac{\mu\Delta h}{2}(1-x_0)^2 + \frac{\mu\Delta h}{2}\Delta^2 + \tilde{x}_1, \;\; x_0 \in [\theta^*, 1-\Delta] \\ \tilde{a}_2(x_0) \equiv -\frac{(1+\mu\Delta)h}{2}(1-x_0)^2 + \frac{(1+\mu\Delta)h}{2}(1-z_1)^2 + \hat{x}_1, \;\; x_0 \in [1-\Delta, \theta+\phi] \\ \tilde{a}_3(x_0) \equiv -\frac{h}{2}(1-x_0)^2 + \frac{h}{2}(1-\theta-\phi)^2 + z, \;\; x_0 \in [\theta+\phi, z_1] \end{cases}$$

$$a'(x_1) = \begin{cases} \frac{1-x_1}{\sqrt{\Delta^2 + \frac{2}{\mu\Delta h}(\tilde{x}_1 - x_1)}}, \;\; x_1 \in [z_1, \tilde{x}_1], \;\; a(z_1) = z \\ \frac{1-x_1}{\sqrt{(1-\theta-\phi)^2 + \frac{2}{(1+\mu\Delta)h}(\hat{x}_1 - x_1)}}, \;\; x_1 \in [\tilde{x}_1, \hat{x}_1], \;\; a(\tilde{x}_1) = \tilde{x}_2 \\ \frac{1-x_1}{\sqrt{(1-z_1)^2 + \frac{2}{h}(z - x_1)}}, \;\; x_1 \in [\hat{x}_1, z], \;\; a(\hat{x}_1) = \hat{x}_2 \end{cases}$$

Assignment function 1: The solution to the given differential equation

$$a'(x_1) = \frac{1-x_1}{\sqrt{\Delta^2 + \frac{2}{\mu\Delta h}(\tilde{x}_1 - x_1)}}$$

with the boundary condition $a(z_1) = z$ and $x_1 \in [z_1, \tilde{x}_1]$ is:

$$\begin{aligned} a(x_1) = \frac{1}{3}\Bigg(& -3\Delta h\mu\sqrt{\frac{\Delta^3 h\mu + 2\tilde{x}_1 - 2x_1}{\Delta h\mu}} + \Delta^4 h^2\mu^2\sqrt{\frac{\Delta^3 h\mu + 2\tilde{x}_1 - 2x_1}{\Delta h\mu}} \\ & + 2\Delta h\mu\tilde{x}_1\sqrt{\frac{\Delta^3 h\mu + 2\tilde{x}_1 - 2x_1}{\Delta h\mu}} + \Delta h\mu x_1\sqrt{\frac{\Delta^3 h\mu + 2\tilde{x}_1 - 2x_1}{\Delta h\mu}} \\ & + 3\Delta h\mu\sqrt{\frac{\Delta^3 h\mu + 2\tilde{x}_1 - 2z_1}{\Delta h\mu}} - \Delta^4 h^2\mu^2\sqrt{\frac{\Delta^3 h\mu + 2\tilde{x}_1 - 2z_1}{\Delta h\mu}} \\ & - 2\Delta h\mu\tilde{x}_1\sqrt{\frac{\Delta^3 h\mu + 2\tilde{x}_1 - 2z_1}{\Delta h\mu}} - \Delta h\mu z_1\sqrt{\frac{\Delta^3 h\mu + 2\tilde{x}_1 - 2z_1}{\Delta h\mu}} \\ & + 3z\Bigg) \end{aligned}$$

Assignment function 2: The general solution to the given differential equation

$$a'(x_1) = \frac{1-x_1}{\sqrt{(1-\theta-\phi)^2 + \frac{2}{(1+\mu\Delta)h}(\hat{x}_1 - x_1)}}$$

is:

$$a(x_1) = \frac{h(1+\Delta\mu)\sqrt{\frac{h(1+\Delta\mu)(-1+\phi+\theta)^2+2(\hat{x}_1-x_1)}{h+\Delta h\mu}}(-3+2\hat{x}_1+h(1+\Delta\mu)(-1+\phi+\theta)^2+x_1)}{3} + C_1$$

Here, C_1 is the constant of integration, which can be determined using the boundary condition $a(\tilde{x}_1) = \tilde{x}_2$. The constant C_1 can be determined using the boundary condition $a(\tilde{x}_1) = \tilde{x}_2$ as follows:

$$\begin{aligned} C_1 = -\frac{1}{3}\Bigg(& h(1+\Delta\mu)\sqrt{\frac{h(1+\Delta\mu)(-1+\phi+\theta)^2+2(\hat{x}_1-\tilde{x}_1)}{h+\Delta h\mu}} \\ & \times (-3+2\hat{x}_1+h(1+\Delta\mu)(-1+\phi+\theta)^2+\tilde{x}_1)\Bigg) \\ & + \tilde{x}_2 \end{aligned}$$

With this constant, you can fully specify the function $a(x_1)$ given the boundary condition. **Assignment function 3:** The solution to the given differential equation

$$a'(x_1) = \frac{1-x_1}{\sqrt{(1-z_1)^2+\frac{2}{h}(z-x_1)}}$$

with the boundary condition $a(\hat{x}_1) = \hat{x}_2$ and $x_1 \in [\hat{x}_1, z]$ is:

$$\begin{aligned} a(x_1) = \frac{1}{3}\Bigg(& 3\hat{x}_2 + 3h\sqrt{\frac{h-2\hat{x}_1-2hz_1+hz_1^2+2z}{h}} - h^2\sqrt{\frac{h-2\hat{x}_1-2hz_1+hz_1^2+2z}{h}} \\ & - h\hat{x}_1\sqrt{\frac{h-2\hat{x}_1-2hz_1+hz_1^2+2z}{h}} + 2h^2z_1\sqrt{\frac{h-2\hat{x}_1-2hz_1+hz_1^2+2z}{h}} \\ & - h^2z_1^2\sqrt{\frac{h-2\hat{x}_1-2hz_1+hz_1^2+2z}{h}} - 2hz\sqrt{\frac{h-2\hat{x}_1-2hz_1+hz_1^2+2z}{h}} \\ & - 3h\sqrt{\frac{h-2x_1-2hz_1+hz_1^2+2z}{h}} + h^2\sqrt{\frac{h-2x_1-2hz_1+hz_1^2+2z}{h}} \\ & + hx_1\sqrt{\frac{h-2x_1-2hz_1+hz_1^2+2z}{h}} - 2h^2z_1\sqrt{\frac{h-2x_1-2hz_1+hz_1^2+2z}{h}} \\ & + h^2z_1^2\sqrt{\frac{h-2x_1-2hz_1+hz_1^2+2z}{h}} + 2hz\sqrt{\frac{h-2x_1-2hz_1+hz_1^2+2z}{h}}\Bigg) \end{aligned}$$

B.4 Machine Management

Assignment function The equilibrium assignment function is the solution to the following equations

$$a'(x_1) = h(1-x_1),\ \ a(\underline{y}) = \theta,\ \ x_1 \in [1-\Delta, \underline{y}] \tag{12}$$

$$a'(x_1) = \frac{1/\Delta}{1/h\Delta + \mu/h_m}(1-x_1),\ \ a(\bar{y}) = \theta + \phi,\ \ x_1 \in [\underline{y}, \bar{y}] \tag{13}$$

$$a'(x_1) = h(1-x_1),\ \ a(z) = 1,\ \ x_1 \in [\bar{y}, z] \tag{14}$$

where the thresholds z, $\underline{y}$, and $\bar{y}$ satisfy the continuity conditions for each interval

$$a(1-\Delta) = z$$
$$a(\underline{y}) = \theta$$
$$a(\bar{y}) = \theta + \phi$$

The equations characterize the assignment function on each of the three intervals. Furthermore, the continuity conditions ensure that the assignment function is continuous on $[1-\Delta, z]$, which is necessary for the resulting assignment function to represent an equilibrium allocation.

$$a_1(x_1) = -\frac{h}{2}(1-x_1)^2 + \frac{h}{2}(1-\underline{y})^2 + \theta,\ \ x_1 \in [1-\Delta, \underline{y}] \tag{15}$$

$$a_2(x_1) = -\frac{\Phi}{2}(1-x_1)^2 + \frac{\Phi}{2}(1-\bar{y})^2 + \theta + \phi,\ \ x_1 \in [\underline{y}, \bar{y}] \tag{16}$$

$$a_3(x_1) = -\frac{h}{2}(1-x_1)^2 + \frac{h}{2}(1-z)^2 + 1,\ \ x_1 \in [\bar{y}, z] \tag{17}$$

where $\Phi \equiv \frac{1/\Delta}{1/h\Delta + \mu/h_m}$. The corresponding employment functions are

$$e(x_2) = 1 - \sqrt{\frac{\theta - x_2 + \frac{h}{2}(1-\underline{y})^2}{h/2}},\ \ x_2 \in [z, \theta]$$

$$e(x_2) = 1 - \sqrt{\frac{\theta + \phi - x_2 + \frac{\Phi}{2}(1-\bar{y})^2}{\Phi/2}},\ \ x_2 \in [\theta, \theta + \phi]$$

$$e(x_2) = 1 - \sqrt{\frac{1 - x_2 + \frac{h}{2}(1-z)^2}{h/2}},\ \ x_2 \in [\theta + \phi, 1]$$

$$
\begin{aligned}
e(x_2) =& 1 - \sqrt{\frac{\theta - x_2}{h/2} + (1-\underline{y})^2}, \;\; x_2 \in [z, \theta] \\
e(x_2) =& 1 - \sqrt{\frac{\theta + \phi - x_2}{\Phi/2} + (1-\bar{y})^2}, \;\; x_2 \in [\theta, \theta + \phi] \\
e(x_2) =& 1 - \sqrt{\frac{1 - x_2}{h/2} + (1-z)^2}, \;\; x_2 \in [\theta + \phi, 1]
\end{aligned}
$$

Thresholds Suppose $\theta > z$. Given the assignment functions solved above, the thresholds z, $\underline{y}$, and $\bar{y}$ satisfy

$$
\begin{aligned}
a(1-\Delta) =& z \\
a(\underline{y}) =& \theta \\
a(\bar{y}) =& \theta + \phi
\end{aligned}
$$

More explicitly,

$$
\begin{aligned}
-\frac{h}{2}\Delta^2 + \frac{h}{2}(1-\underline{y})^2 + \theta = z \\
-\frac{\Phi}{2}(1-\underline{y})^2 + \frac{\Phi}{2}(1-\bar{y})^2 + \theta + \phi = \theta \\
-\frac{h}{2}(1-\bar{y})^2 + \frac{h}{2}(1-z)^2 + 1 = \theta + \phi
\end{aligned}
$$

where $\Phi \equiv \frac{1/\Delta}{1/h\Delta + \mu/h_m}$. The second equation is

$$
\begin{aligned}
-\frac{\Phi}{2}(1-\underline{y})^2 + \phi = -\frac{\Phi}{2}(1-\bar{y})^2 \\
(1-\underline{y})^2 - \phi\frac{2}{\Phi} = (1-\bar{y})^2
\end{aligned}
$$

Using the first equation,

$$
(1-\underline{y})^2 = z\frac{2}{h} + \Delta^2 - \frac{2}{h}\theta
$$

$$
\begin{aligned}
(1-\bar{y})^2 =&(1-\underline{y})^2 - \phi\frac{2}{\Phi} \\
=&z\frac{2}{h} + \Delta^2 - \frac{2}{h}\theta - \phi\frac{2}{\Phi}
\end{aligned}
$$

The third equation is then

$$
-\frac{h}{2}\left[z\frac{2}{h} + \Delta^2 - \frac{2}{h}\theta - \phi\frac{2}{\Phi}\right] + \frac{h}{2}(1-z)^2 + 1 - \theta - \phi = 0
$$

Since $z < 1$, it follows that

$$
z = 1 + 1/h - \sqrt{1/h^2 + \Delta^2 + 2\phi(1/h - 1/\Phi)}
$$

Given the value of z, $\underline{y}$ and $\bar{y}$ are

$$
\begin{aligned}
1-\underline{y} =&\sqrt{z\frac{2}{h} + \Delta^2 - \frac{2}{h}\theta} \\
1-\bar{y} =&\sqrt{z\frac{2}{h} + \Delta^2 - \frac{2}{h}\theta - \phi\frac{2}{\Phi}}
\end{aligned}
$$

$$
\begin{aligned}
\underline{y} =&1 - \sqrt{z\frac{2}{h} + \Delta^2 - \frac{2}{h}\theta} \\
\bar{y} =&1 - \sqrt{z\frac{2}{h} + \Delta^2 - \frac{2}{h}\theta - \phi\frac{2}{\Phi}}
\end{aligned}
$$

Wages The equilibrium wage function is determined by the first-order condition

$$
w_1'(x_1) = \frac{x_2 - w_1(x_1)}{1 - x_1}
$$

The general solution to the above differential equation is

$$
w_1(x_1) = \begin{cases}
w_{11}(x_1) = a_1(x_1) - hx_1(1-x_1) + C_1(1-x_1), & x_1 \in [1-\Delta, \underline{y}] \\
w_{12}(x_1) = a_2(x_1) - \Phi x_1(1-x_1) + C_2(1-x_1), & x_1 \in [\underline{y}, \bar{y}] \\
w_{13}(x_1) = a_3(x_1) - hx_1(1-x_1) + C_3(1-x_1), & x_1 \in [\bar{y}, z]
\end{cases}
$$

where $\Phi \equiv \frac{1/\Delta}{1/h\Delta + \mu/h_m}$ and C_k, $k = 1, 2, 3$, is some constant. Again, Workers' wages must be continuous at $1 - \Delta$ and z. Moreover, the marginal managers must earn the same amount as the marginal workers. Thus, the constants of integration must satisfy

$$\begin{aligned}
a(\underline{y}) - h\underline{y}(1-\underline{y}) + C_1(1-\underline{y}) =& a(\underline{y}) - \Phi\underline{y}(1-\underline{y}) + C_2(1-\underline{y}) \\
a(\bar{y}) - \Phi\bar{y}(1-\bar{y}) + C_2(1-\bar{y}) =& a(\bar{y}) - h\bar{y}(1-\bar{y}) + C_3(1-\bar{y}) \\
a(z) - hz(1-z) + C_3(1-z) =& w_2(z) = \frac{z - w_1(1-\Delta)}{h\Delta}
\end{aligned}$$

Rearrange the terms to obtain

$$\begin{aligned}
-h\underline{y}(1-\underline{y}) + C_1(1-\underline{y}) =& -\Phi\underline{y}(1-\underline{y}) + C_2(1-\underline{y}) \\
-\Phi\bar{y}(1-\bar{y}) + C_2(1-\bar{y}) =& -h\bar{y}(1-\bar{y}) + C_3(1-\bar{y}) \\
1 - hz(1-z) + C_3(1-z) =& \frac{h(1-\Delta)\Delta - C_1\Delta}{h\Delta}
\end{aligned}$$

$$\begin{aligned}
C_1 =& h\underline{y} - \Phi\underline{y} + C_2 \\
C_2 =& \Phi\bar{y} - h\bar{y} + C_3 \\
1 - hz(1-z) + C_3(1-z) =& 1 - \Delta - \frac{C_1}{h}
\end{aligned}$$

Combine the first and second equations

$$\begin{aligned}
C_1 =& h\underline{y} - \Phi\underline{y} + \Phi\bar{y} - h\bar{y} + C_3 \\
=& C_3 + (h-\Phi)\underline{y} - (h-\Phi)\bar{y} \\
=& C_3 - (h-\Phi)(\bar{y} - \underline{y})
\end{aligned}$$

Substituting the above into the third equation, I obtain

$$\begin{aligned}
1 - hz(1-z) + C_3(1-z) =& 1 - \Delta - \frac{1}{h}C_3 + \frac{1}{h}(h-\Phi)(\bar{y} - \underline{y}) \\
C_3 =& \frac{hz(1-z) - \Delta + (h-\Phi)(\bar{y} - \underline{y})/h}{1 - z + 1/h}
\end{aligned}$$

B.5 Machines as Middle Managers

The analysis so far is based on the organizational structure where machines are at the top of the hierarchy. This setup captures the long-term case where machines have the ability to run organizations on par with human managers.

However, even without the ability to run an entire organization, machines today have already automated various managerial tasks, as mentioned above. Thus, machines today are competing with middle managers who supervise workers but do not run entire organizations. In order to examine this case, I modify the setup and analyze an equilibrium with machines in the middle layer. Consider the following assignment equations

$$f(x_1) = \left[\frac{1}{h(1-x_1)} \cdot f\left(a(x_1)\right) + \frac{1}{h_m(1-x_1)} \cdot g(a(x_1))\right] a'(x_1)$$

$$f(x_2) + g(x_2) = \frac{1}{h(1-x_2)} \cdot f\left(a(x_2)\right) a'(x_2)$$

The first equation links workers and middle managers, and is analogous to (5). The second equation links the middle managers and top managers. The equations assume that machines are only in the middle layer. As before, I solve this setup and verify that there is indeed an equilibrium that justifies the allocation.

With machines as middle managers, technological change may have the opposite effect on top income. Since it is the middle managers who are competing with machines, only the least skilled managers who are in the second layer experience a decline in their wages. On the other hand, top managers who supervise the middle managers are complemented by machines because they are in the upper layer.